

HEMISPHERICAL SHELLS SUBJECTED TO  
DYNAMIC, THERMAL, AND RANDOM LOADINGS

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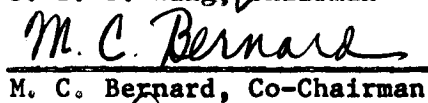
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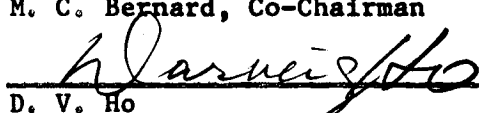
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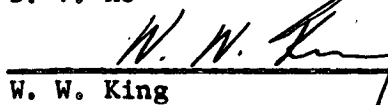
HEMISPHERICAL SHELLS SUBJECTED TO  
DYNAMIC, THERMAL, AND RANDOM LOADINGS

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This thesis is dedicated to Iris.

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## SYMBOLS

$a$	Hemispherical shell radius
$b$	Loading parameter
$D$	Shell flexural rigidity
$E$	Modulus of elasticity
$h$	Shell thickness
$l$	Length of cylindrical shell
$M_\phi, M_\theta, M_{\theta\phi}, M_{\phi\theta}$	Stress couples
$N_\phi, N_\theta, N_{\theta\phi}, N_{\phi\theta}$	Stress resultants
$M_o$	Edge bending moment
$P_\alpha$	Loading components
$v_\phi, v_\theta$	Transverse shear along meridional line
$t$	Time
$u_\phi, u_\theta, w$	Components of displacements
$\phi, \theta, \zeta$	Spherical coordinates
$\epsilon_\phi^o, \epsilon_\theta^o, \gamma_{\phi\theta}^o, \gamma_{\theta\phi}^o$	Components of Strain
$\epsilon_\zeta, \gamma_{\phi\zeta}^o, \gamma_{\theta\zeta}^o$	
$K_\phi, K_\theta, \kappa_\phi, \kappa_\theta$	Components of change of curvature
$\nu$	Poisson's ratio
$\rho$	Mass density
$\omega, \Omega, \omega_{in}$	Frequencies
$\langle \rangle$	Ensemble average
$\{ \}$	Random process

$\Gamma$	Correlation function
$G$	Power spectral density
RMS	Root-mean-square response
Superscript (K)	Index for Kth element of ensemble
$Q_o$	Transverse shear component at junction of cylindrical shell and bulkhead
$K_s$	Averaging coefficient for the shear
$K_d$	Thermal diffusivity
$K_c$	Thermal conductivity
$\bar{e}_\phi, \bar{e}_\theta, \bar{e}_n$	Base vectors
$\alpha_o$	Coefficient of linear thermal expansion
$\beta_\phi, \beta_\theta$	Changes of slope of the normal to the middle surface
$\theta^*$	Temperature
$Q_o^*$	Heat flux
$B^* = \left( \frac{Dh^3}{\rho K_d^2 a^4} \right)^{1/4}$	Inertia parameter

## SUMMARY

The thesis is a study of the dynamic aspects of a hemispherical shell with various edge conditions and subjected to various types of loading.

By using the modal analysis, closed-form solutions are obtained for the axisymmetric dynamic response of hemispherical shells with roller-hinged edges and roller-clamped edges. By applying a harmonic ring load to withhold the motion either along the longitudinal direction of the roller-hinged edge or along the transverse direction of the roller-clamped edge, the eigenfunctions for the hinged and clamped edges, respectively, are obtained. Numerical results for free vibrations and the dynamic response of the shell are obtained and discussed in detail. Finally, it is shown that this analysis can equally well be carried out by applying the mode-acceleration method of Williams.

Although the analysis carried out for a problem with external dynamic loading is similar to the analysis of a problem with thermal loading, the inhomogeneous boundary conditions in the displacements arising from the thermal effect must be considered in the latter problem. To take care of this inhomogeneity, a particular term is added to the formal solution so that the resulting solution satisfies the governing differential equations and the boundary conditions.

To provide an example of the application of this procedure to hemispherical shells, the free vibration of an elastic cylinder with a

hemispherical shell bottom is studied. Comparison made with respect to a clamped-clamped cylinder shows that the effect of the bottom is larger for thicker cylinders and particularly significant when the cylinder is short.

Finally, an investigation of the random excitation of thin elastic shells is made. The problem is discussed in detail for hemispherical shells with roller-hinged and roller-clamped edges. Numerical results are obtained for a hemispherical shell with a roller-clamped edge when the shell is subjected to a non-stationary separable random process uniformly distributed over the shell surface. Both wide-band and band-limited power spectral densities are included.

## CHAPTER I

### INTRODUCTION

By using the inextensional theory to investigate the acoustic behavior of bells, the vibration of thin spherical shells was first studied by Lord Rayleigh [1, 2] in 1881. In 1882, Lamb [3] first determined the natural frequencies of a closed spherical monocoque shell. Love [4] in 1888 developed the general theory for small free vibrations of thin elastic spherical shells. With the use of a technique introduced by Van der Neut [5], the equations of the classical dynamic bending theory of elastic spherical shells was derived by Federhofer [6]. Using essentially the same technique, uncoupled equations which include the effects of transverse shear deformation and rotatory inertia have been derived by Kalnins [7] for shallow spherical shells and extended by Prasad [8] to nonshallow spherical shells. By using the approach from Berry [9], free axisymmetric vibrations were studied by Naghdi and Kalnins [10], where the natural frequencies for free-edged hemispherical shells with thickness-radius ratios larger than 0.01 have been obtained. The frequency equations corresponding to spherical shells and hemispherical shells are discussed by Kalnins in [11]. Hwang [12] obtained the natural frequencies for a hemispherical shell using a method similar to that used by Baker [13] to obtain the frequencies for a complete spherical

shell, neglecting the bending effects. In [14] Kalnins presents a numerical method for the calculation of the natural frequencies and normal modes of arbitrary rotationally symmetric shells. The non-symmetric dynamic problems of elastic spherical shells have been studied by Silbiger [15] and then by Wilkinson and Kalnins [16]. Baker, Hu, and Jackson [17] have studied the axisymmetrically dynamic response of a complete spherical shell by using, basically, the membrane theory. Hwang [18] has obtained the experimental results of a thin hemispherical shell having a free edge.

The present study is concerned with the dynamic aspects of a hemispherical shell with various edge conditions and subjected to dynamic, thermal and random loadings.

In order to derive the set of governing equations in a more complete fashion, the assumption of seven stress-displacement relations has been used. The variational theorem of the energy functional yields the stress differential equations of motion, which include the rotatory inertia terms and the effect from a visco-elastic foundation, and ten stress-displacement relations, which take into account the transverse shear deformation and a thermal input.

Solutions corresponding to various boundary conditions are sought for the axisymmetric response to a dynamic load and to a thermal load. The results may be used for many practical problems such as the response of a hemispherical nose of a space vehicle during its re-entry when it is subjected to both a high temperature gradient and an atmospheric pressure acting on the shell surface. Furthermore,

the free vibration of an elastic cylinder with an elastic hemispherical bottom is investigated. Finally, the response of a hemispherical shell to a random excitation is investigated by use of a modal analysis.

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## CHAPTER II

### BASIC EQUATIONS FOR SPHERICAL SHELLS

To derive a set of governing equations for spherical shells that are more complete than those usually used the seven components of strain proposed by Reissner [1] are used. A linear distribution of  $\Omega^0$ , which is defined to be the rotation of the surface element around the normal, has been assumed instead of the constant distribution used in [1]. Furthermore, the potential energy from a visco-elastic foundation, the kinetic energy, and the thermal strain energy of the shell have been added to the energy functional considered by Reissner [2] and Naghdi [3].

The geometry and sign conventions of the shell are shown in Figure 1. The displacement of a point in space has the form

$$\bar{U} = U_{\phi} \bar{e}_{\phi} + U_{\theta} \bar{e}_{\theta} + W \bar{e}_n \quad (2.1)$$

where  $\bar{e}_{\phi}$ ,  $\bar{e}_{\theta}$ , and  $\bar{e}_n$  are base vectors and  $U_{\phi}$ ,  $U_{\theta}$ , and  $W$  are displacements along the directions of the base vectors.

Using the orthogonal curvilinear coordinate system  $\phi$ ,  $\theta$ , and  $\zeta$ , where  $\phi$  and  $\theta$  are the angular spherical coordinates of a point on the middle surface of the shell and  $\zeta$  is the distance measured along the outward normal from the middle surface the seven components of strain are deduced from [1]. They are

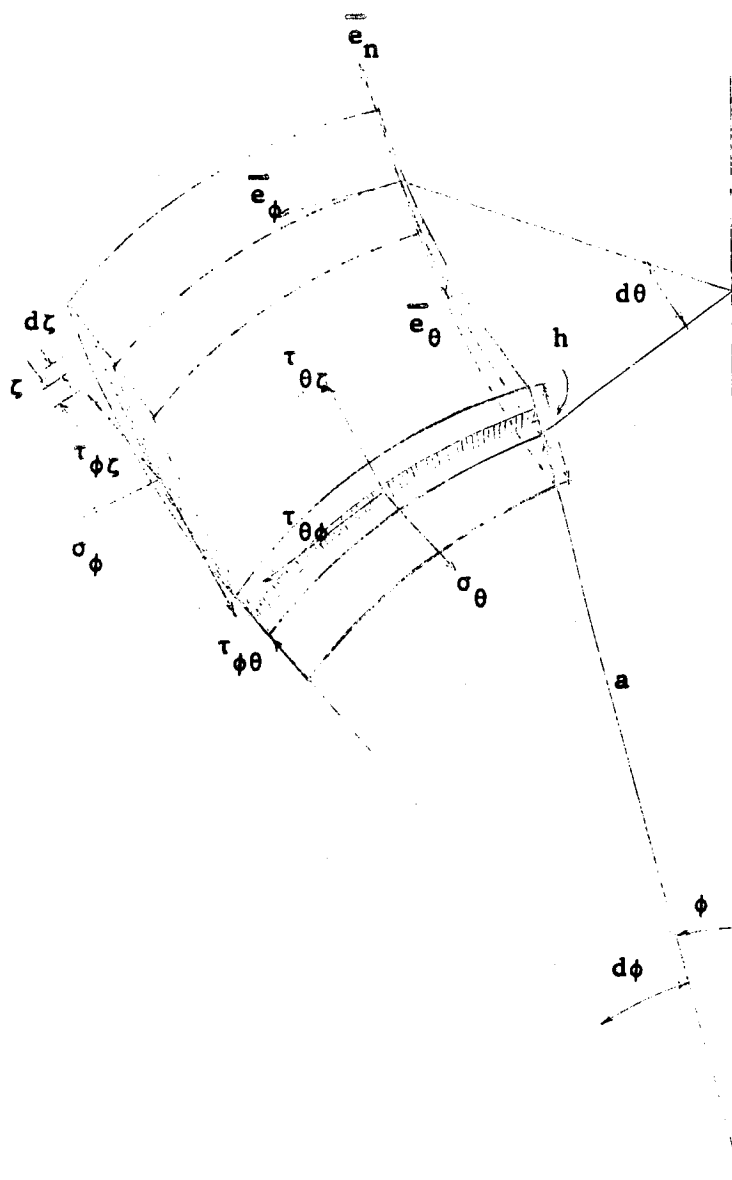


Figure 1. Geometry and Sign Conventions of the Shell

$$\epsilon_{\phi} = \frac{1}{aA_{\zeta}} \left\{ \frac{\partial U_{\phi}}{\partial \phi} + W \right\} \quad (2.2)$$

$$\epsilon_{\theta} = \frac{1}{aA_{\zeta}} \left\{ \frac{1}{\sin \phi} \frac{\partial U_{\theta}}{\partial \theta} + U_{\phi} \cot \phi + W \right\}$$

$$\gamma_{\phi\theta} = \frac{1}{aA_{\zeta}} \frac{\partial U_{\theta}}{\partial \phi} + \Omega^{\circ}$$

$$\gamma_{\theta\phi} = \frac{1}{aA_{\zeta}} \left\{ \frac{1}{\sin \phi} \frac{\partial U_{\phi}}{\partial \theta} - U_{\theta} \cot \phi \right\} - \Omega^{\circ}$$

$$\epsilon_{\zeta} = + \frac{\partial W}{\partial \zeta}$$

$$\gamma_{\phi\zeta} = + \frac{1}{aA_{\zeta}} \frac{\partial W}{\partial \phi} + A_{\zeta} \frac{\partial}{\partial \zeta} \left( \frac{U_{\phi}}{A_{\zeta}} \right)$$

$$\gamma_{\theta\zeta} = + \frac{1}{aA_{\zeta}} \frac{1}{\sin \phi} \frac{\partial W}{\partial \theta} + A_{\zeta} \frac{\partial}{\partial \zeta} \left( \frac{U_{\theta}}{A_{\zeta}} \right)$$

Here  $a$  is the principal radius of curvature of the middle surface of the spherical shell,  $\Omega^{\circ}$  the rotation of the surface element around the normal, and

$$A_{\zeta} = 1 + \frac{\zeta}{a} \quad (2.3)$$

The displacements are approximated in the following manner:

$$U_{\phi} = u_{\phi} + \zeta \beta_{\phi} \quad (2.4)$$

$$U_{\theta} = u_{\theta} + \zeta \beta_{\theta}$$

$$W = w + \zeta w' + \frac{1}{2} \zeta^2 w''$$

$$\Omega^{\circ} = \omega^{\circ} + \zeta \omega'$$

In equations (2.4)  $\beta_{\phi}$  and  $\beta_{\theta}$  are the changes of slope of the normal to the middle surface along the coordinates  $\phi$  and  $\theta$ , respectively,  $u_{\phi}$ ,  $u_{\theta}$ , and  $w$  are the components of displacements of a point on the middle surface, and  $\omega^{\circ}$  is the rotation of the middle surface around the normal. Primes indicate differentiation with respect to  $\zeta$ .

By substituting equations (2.4) into equations (2.2) one obtains

$$\epsilon_{\phi} = \frac{1}{A_{\zeta}} \left\{ \epsilon_{\phi}^{\circ} + \zeta K_{\phi}^* + \frac{1}{2} \zeta^2 \frac{w''}{a} \right\} \quad (2.5)$$

$$\epsilon_{\theta} = \frac{1}{A_{\zeta}} \left\{ \epsilon_{\theta}^{\circ} + \zeta K_{\theta}^* + \frac{1}{2} \zeta^2 \frac{w''}{a} \right\}$$

$$\gamma_{\phi\theta} = \frac{1}{A_{\zeta}} \{ \gamma_{\phi}^{\circ} + \zeta \kappa_{\phi} \} + \omega^{\circ} + \zeta \omega'$$

$$\gamma_{\theta\phi} = \frac{1}{A_{\zeta}} \{ \gamma_{\theta}^{\circ} + \zeta \kappa_{\theta} \} - \omega^{\circ} - \zeta \omega'$$

$$\varepsilon_{\zeta} = + w' + \zeta w''$$

$$\gamma_{\phi\zeta} = \gamma_{\phi\zeta}^{\circ} + \frac{\zeta}{a} \left\{ \frac{\partial w'}{\partial \phi} + \frac{\zeta}{2} \frac{\partial w''}{\partial \phi} \right\}$$

$$\gamma_{\theta\zeta} = \gamma_{\theta\zeta}^{\circ} + \frac{\zeta}{a \sin \phi} \left\{ \frac{\partial w'}{\partial \theta} + \frac{\zeta}{2} \frac{\partial w''}{\partial \theta} \right\}$$

where

$$\varepsilon_{\phi}^{\circ} = \frac{1}{a} \left\{ \frac{\partial u_{\phi}}{\partial \phi} + w \right\}$$

(2.6)

$$\varepsilon_{\theta}^{\circ} = \frac{1}{a} \left\{ \frac{1}{\sin \phi} \frac{\partial u_{\theta}}{\partial \theta} + u_{\phi} \cot \phi + w \right\}$$

$$\gamma_{\phi}^{\circ} = \frac{1}{a} \frac{\partial u_{\theta}}{\partial \phi}$$

$$\gamma_{\theta}^{\circ} = \frac{1}{a \sin \phi} \left\{ \frac{\partial u_{\phi}}{\partial \theta} - u_{\theta} \cos \phi \right\}$$

$$\gamma_{\phi\zeta}^{\circ} = + \frac{1}{a} \frac{\partial w}{\partial \phi} - \frac{u_{\phi}}{a} + \beta_{\phi}$$

$$\gamma_{\theta\zeta}^{\circ} = + \frac{1}{a \sin \phi} \frac{\partial w}{\partial \theta} - \frac{u_{\theta}}{a} + \beta_{\theta}$$

and

$$K_{\phi}^* = K_{\phi} + \frac{w'}{a} \quad (2.7)$$

$$K_{\theta}^* = K_{\theta} + \frac{w'}{a}$$

$$K_{\phi} = \frac{1}{a} \cdot \frac{\partial \beta_{\phi}}{\partial \phi}$$

$$K_{\theta} = \frac{1}{a \sin \phi} \left\{ \frac{\partial \beta_{\theta}}{\partial \theta} + \beta_{\phi} \cos \phi \right\}$$

$$\kappa_{\phi} = \frac{1}{a} \cdot \frac{\partial \beta_{\theta}}{\partial \phi}$$

$$\kappa_{\theta} = \frac{1}{a \sin \phi} \left\{ \frac{\partial \beta_{\phi}}{\partial \theta} - \beta_{\theta} \cos \phi \right\}$$

The forces and moments per unit length on the middle surface are of the following form:

$$(N_{\phi}, N_{\phi\theta}, V_{\phi}) = \int_{-h/2}^{h/2} (\sigma_{\phi}, \tau_{\phi\theta}, \tau_{\phi z}) A_z dz \quad (2.8)$$

$$(N_{\theta\phi}, N_{\theta}, V_{\theta}) = \int_{-h/2}^{h/2} (\tau_{\theta\phi}, \sigma_{\theta}, \tau_{\theta z}) A_z dz$$

$$(M_{\phi}, M_{\phi\theta}) = \int_{-h/2}^{h/2} (\sigma_{\phi}, \tau_{\phi\theta}) z A_z dz$$



$$(M_{\theta\phi}, M_{\theta}) = \int_{-h/2}^{h/2} (\tau_{\theta\phi}, \tau_{\theta}) \zeta A_{\zeta} d\zeta$$

where  $\sigma_{\phi}$ ,  $\sigma_{\theta}$  are the normal stresses,  $\tau_{\phi\theta}$ ,  $\tau_{\phi\zeta}$ ,  $\tau_{\theta\zeta}$  the shearing stresses,  $N_{\phi}$ ,  $N_{\theta}$ ,  $N_{\phi\theta}$  and  $N_{\theta\phi}$  the stress resultants,  $M_{\phi}$ ,  $M_{\theta}$ ,  $M_{\phi\theta}$  and  $M_{\theta\phi}$  the stress couples, and  $V_{\phi}$  and  $V_{\theta}$  the transverse shear stress resultants.

As usual, the components of stress are considered to vary linearly throughout the thickness of the shell. That is

$$\begin{bmatrix} \sigma_{\phi} \\ \sigma_{\theta} \\ \tau_{\phi\theta} \\ \tau_{\theta\phi} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \zeta \quad (2.9a)$$

and

$$\tau_{\phi\zeta} = \frac{a_5}{A_{\zeta}} V_{\phi} \left[ 1 - \left( \frac{\zeta}{h/2} \right)^2 \right] - \frac{1}{4} \left\{ p_{\phi}^{+} \cdot H_{\phi}^{+} + p_{\phi}^{-} \cdot H_{\phi}^{-} \right\} \quad (2.9b)$$

$$\tau_{\theta\zeta} = \frac{a_5}{A_{\zeta}} V_{\theta} \left[ 1 - \left( \frac{\zeta}{h/2} \right)^2 \right] - \frac{1}{4} \left\{ p_{\theta}^{+} \cdot H_{\theta}^{+} + p_{\theta}^{-} \cdot H_{\theta}^{-} \right\}$$

$$\sigma_{\zeta} = \frac{k_n}{A_{\zeta}^2} \left\{ \left[ \frac{S_1}{h} + \frac{S_2}{h^2} \zeta \right] \left[ 1 - \left( \frac{\zeta}{h/2} \right)^2 \right] \right\}$$

$$+ \frac{1}{2} \left[ p_n^+ H_n^+ + p_n^- H_n^- \right] \left. \right\}$$

where  $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4$  are to be determined by using equations (2.8), and the functions  $S_1$  and  $S_2$  are still undetermined.  $p_\phi^+$  and  $p_\phi^-$ ,  $p_\theta^+$  and  $p_\theta^-$ , and  $p_n^+$  and  $p_n^-$  are, respectively, the values of  $\tau_{\phi\zeta}$ ,  $\tau_{\theta\zeta}$ , and  $\sigma_\zeta$  at the outer and inner surfaces of the shell. The coefficient  $k_n$  is a distinguishing factor between the contribution of transverse shear deformation and normal stress.

Finally

$$\begin{aligned} H_\theta^+ &= H_\phi^+ = \left\{ 1 - 2 \frac{\zeta}{h/2} - 3 \left( \frac{\zeta}{h/2} \right)^2 \right\} \left( 1 + \frac{h}{2a} \right) \frac{1}{A_\zeta} \\ H_\theta^- &= H_\phi^- = \left\{ 1 + 2 \frac{\zeta}{h/2} - 3 \left( \frac{\zeta}{h/2} \right)^2 \right\} \left( 1 - \frac{h}{2a} \right) \frac{1}{A_\zeta} \\ H_n^+ &= \left\{ 1 + \frac{3}{2} \left( \frac{\zeta}{h/2} \right) - \frac{1}{2} \left( \frac{\zeta}{h/2} \right)^3 \right\} \left[ 1 + \frac{h}{2a} \right]^2 \frac{1}{A_\zeta^2} \\ H_n^- &= \left\{ 1 - \frac{3}{2} \left( \frac{\zeta}{h/2} \right) + \frac{1}{2} \left( \frac{\zeta}{h/2} \right)^3 \right\} \left[ 1 - \frac{h}{2a} \right]^2 \frac{1}{A_\zeta^2} \end{aligned} \quad (2.9c)$$

are obtained by using the stress equilibrium equations of elasticity in the  $\zeta$  plane together with the prescribed boundary conditions for  $\tau_{\phi\zeta}$ ,  $\tau_{\theta\zeta}$ , and  $\sigma_\zeta$  [4]\*.

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\*Page 93 of reference [4].

By substituting equations (2.9a), (2.9b), (2.9c) into equations (2.8),  $a_1 \dots a_5$ , and  $b_1 \dots b_4$  are obtained as follows:

$$\begin{array}{ccc} a_1 & N_\phi & b_1 & M_\phi \\ a_2 & = \frac{1}{h} & N_\theta & b_2 & = \frac{12}{h^3} & M_\theta \\ a_3 & N_{\phi\theta} & b_3 & M_{\phi\theta} \\ a_4 & N_{\theta\phi} & b_4 & M_{\theta\phi} \\ a_5 & 3/2 & & \end{array} \quad (2.10)$$

With the rotary inertia, energy from a viscoelastic foundation, and thermal strain energy added, the variation of the energy functional in the action integral form is

$$\begin{aligned} \delta A = & \delta \int_{t_0}^{t_1} \iiint_V \left( \sigma_\phi \epsilon_\phi + \sigma_\theta \epsilon_\theta + \sigma_\zeta \epsilon_\zeta + \tau_{\phi\theta} \gamma_{\phi\theta} + \tau_{\theta\phi} \gamma_{\theta\phi} \right. \\ & \left. + \tau_{\phi\zeta} \gamma_{\phi\zeta} + \tau_{\theta\zeta} \gamma_{\theta\zeta} \right) - \frac{\rho}{2} \left[ \left( \frac{\partial U_\phi}{\partial t} \right)^2 + \left( \frac{\partial U_\theta}{\partial t} \right)^2 + \left( \frac{\partial W}{\partial t} \right)^2 \right. \\ & \left. - \Gamma_0 \right] A_\zeta^2 a^2 \sin \phi \, d\phi d\theta d\zeta dt \\ & - \delta \int_{t_0}^{t_1} \iint_S \left[ -\frac{k_f}{2} (u_\phi^2 + u_\theta^2 + w^2) + (p_\phi^+ U_\phi^+ + p_\theta^+ U_\theta^+ \right. \\ & \left. + p_n^+ W^+) \left( 1 + \frac{h}{2a} \right)^2 + (p_\phi^- U_\phi^- + p_\theta^- U_\theta^- + p_n^- W^-) \left( 1 - \frac{h}{2a} \right)^2 \right] \end{aligned} \quad (2.11)$$

$$\begin{aligned}
& \cdot a^2 \sin \phi d\phi d\theta dt = \delta \int_{t_0}^{t_1} \oint \int_{-h/2}^{h/2} (\sigma_n U_n + \tau_{nt} U_t \\
& + \tau_{n\zeta} W) \left(1 + \frac{\zeta}{a}\right) d\zeta \alpha_t d\xi_t dt + \int_{t_0}^{t_1} \iint_S \lambda_f \\
& \cdot \left[ \frac{\partial u_\phi}{\partial t} \delta u_\phi + \frac{\partial u_\theta}{\partial t} \delta u_\theta + \frac{\partial w}{\partial t} \delta w \right] a^2 \sin \phi d\phi d\theta dt
\end{aligned}$$

where  $V$  denotes the volume,  $t$  is the time,  $S$  indicates that part of the surface where the surface loads  $p_\phi^+$ ,  $p_\theta^+$ ,  $p_n^+$ ,  $p_\phi^-$ ,  $p_\theta^-$ ,  $p_n^-$  are prescribed,  $U_\phi^+$ ,  $U_\theta^+$ ,  $W^+$ ,  $U_\phi^-$ ,  $U_\theta^-$ ,  $W^-$  designate the displacements  $U_\phi$ ,  $U_\theta$ , and  $W$  at the outer and inner surfaces, respectively, and  $k_f$  and  $\lambda_f$  are the elastic parameter and viscous damping parameter, respectively, for the foundation. It has been assumed that  $k_f$  and  $\lambda_f$  are the same in the three directions  $\phi$ ,  $\theta$  and  $\zeta$ . The third integral represents the potential of the edge loads and the subscripts  $n$  and  $t$  refer to the normal and tangential directions on the boundary faces. The stress resultants and the stress couples due to edge stresses,  $N_n$ ,  $N_{nt}$ ,  $V_n$ ,  $M_n$ , and  $M_{nt}$ , are defined similarly to those in equations (2.8). The last integral represents the non-conservative energy due to the damping of the foundation\*. We note that

$$Q_n = \int_{-h/2}^{h/2} \tau_{n\zeta} A_\zeta a^2 d\zeta \quad (2.12)$$

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\*Page 56 of reference [5].

and furthermore that\*\*

$$\begin{aligned} \Gamma_0 = \frac{1}{2E} & \left[ \sigma_\phi^2 + \sigma_\theta^2 + \sigma_\zeta^2 - 2\nu (\sigma_\phi \sigma_\theta + \sigma_\phi \sigma_\zeta + \sigma_\theta \sigma_\zeta) \right. \\ & \left. + 2(1+\nu) (\tau_{\phi\theta}^2 + \tau_{\theta\phi}^2 + \tau_{\phi\zeta}^2 + \tau_{\theta\zeta}^2) \right] \\ & + \alpha_0 \theta^* (\sigma_\phi + \sigma_\theta + \sigma_\zeta) \end{aligned} \quad (2.13)$$

is the complementary-energy density for Hookean materials, where  $t$  denotes time,  $\alpha_0$  the coefficient of linear expansion,  $\theta^*$  the temperature,  $E$  the Young's modulus, and  $\nu$  is the Poisson's ratio.

Equation (2.11) is now integrated with respect to  $\zeta$  throughout the thickness of the shell. Next the variation indicated in equation (2.11) is performed. Then the coefficients of the variational changes in the deformations, stress resultants and stress couples are set equal to zero\*\*\*. If quadratic and higher order terms in  $h/a$  are neglected, in the case where  $\sigma_\zeta = 0$  and  $W = w$  a system of equations governing the behavior of spherical shells is obtained. They may be grouped in two parts. First there are the following relations:

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\*\*Reference [6], page 124.

\*\*\*Principle of stationary action, pp. 159-162 of reference [7].

$$N_{\phi} = \frac{Eh}{1-\nu^2} \left[ \epsilon_{\phi}^o + \nu \epsilon_{\phi}^o \right] - \frac{E\alpha_0 \theta_0^*}{1-\nu} \quad (2.14)$$

$$N_{\theta} = \frac{Eh}{1-\nu^2} \left[ \epsilon_{\theta}^o + \nu \epsilon_{\phi}^o \right] - \frac{F\alpha_0 \theta_0^*}{1-\nu}$$

$$N_{\phi\theta} = \frac{Eh}{1+\nu} \left[ \gamma_{\phi}^o + \omega^o + a\alpha\omega' \right]$$

$$N_{\theta\phi} = \frac{Eh}{1+\nu} \left[ \gamma_{\theta}^o - \omega^o - a\alpha\omega' \right]$$

$$M_{\phi} = D \left[ K_{\phi} + \nu K_{\theta} \right] - \frac{E\alpha_0 \theta_1^*}{1-\nu}$$

$$M_{\theta} = D \left[ K_{\theta} + \nu K_{\phi} \right] - \frac{E\alpha_0 \theta_1^*}{1-\nu}$$

$$M_{\phi\theta} = \frac{Eh^3}{12(1+\nu)} \left[ \kappa_{\phi} + \omega^o/a + \omega' \right]$$

$$M_{\theta\phi} = \frac{Eh^3}{12(1+\nu)} \left[ \kappa_{\theta} - \omega^o/a - \omega' \right]$$

$$v_{\phi} = \frac{5}{6} Gh \left[ \gamma_{\phi\zeta}^o \right] + \frac{m_{\phi}}{6}$$

$$v_{\theta} = \frac{5}{6} Gh \left[ \gamma_{\theta\zeta}^o \right] + \frac{m_{\theta}}{6}$$

with

$$\alpha = \frac{1}{12} (h/a)^2, \quad \theta_0^* = \int_{-h/2}^{h/2} \theta^* d\zeta, \quad \theta_1^* = \int_{-h/2}^{h/2} \theta^* \zeta d\zeta \quad (2.15)$$

$$m_\phi = \frac{h}{2} (p_\phi^+ H_\phi^+ + p_\phi^- H_\phi^-), \quad m_\theta = \frac{h}{2} (p_\theta^+ H_\theta^+ + p_\theta^- H_\theta^-)$$

and

$$\omega^0 = \frac{1}{2(1-\alpha)} \left[ (\gamma_\theta^0 - \gamma_\phi^0) - \alpha a (\kappa_\theta - \kappa_\phi) \right] \quad (2.16)$$

$$\omega' = \frac{1}{2(1-\alpha)} \left[ (\kappa_\theta - \kappa_\phi) - 1/a (\gamma_\theta^0 - \gamma_\phi^0) \right]$$

Next there are the equations of motion. They are

$$\frac{\partial N}{\partial \phi} + \frac{\partial N}{\partial \phi} \text{Cosec } \phi + (N_\phi - N_\theta) \text{Cot } \phi + V_\phi \quad (2.17)$$

$$= a \left( \rho h k_1 \frac{\partial^2 u}{\partial t^2} + \rho h a k_2 \frac{\partial^2 \beta}{\partial t^2} + k_f u_\phi + \lambda_f \frac{\partial u}{\partial t} \right) - a p_\phi$$

$$\frac{\partial N}{\partial \phi} + \frac{\partial N}{\partial \theta} \text{Cosec } \phi + N_{\phi\theta} \text{Cot } \phi + V_\theta + N_{\theta\phi} \text{Cot } \phi$$

$$= a \left( \rho h k_1 \frac{\partial^2 u}{\partial t^2} + \rho h a k_2 \frac{\partial^2 \beta_\theta}{\partial t^2} + k_f u_\theta + \lambda_f \frac{\partial u_\theta}{\partial t} \right) - a p_\theta$$

$$\frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_\theta}{\partial \theta} \operatorname{Cosec} \phi + V_\phi \cot \phi - (N_\phi + N_\theta)$$

$$= a \left( \rho h k_1 \frac{\partial^2 w}{\partial t^2} + k_f w + \lambda_f \frac{\partial w}{\partial t} \right) - a p_n$$

$$\frac{\partial M_\phi}{\partial \phi} + \frac{\partial M_{\theta\phi}}{\partial \theta} \operatorname{Cosec} \phi + (M_\phi - M_\theta) \cot \phi - a(V_\phi - m_\phi)$$

$$= \frac{1}{12} \rho h^3 a \left( k_r \frac{\partial^2 \beta_\phi}{\partial t^2} + \frac{c_r}{a} \frac{\partial^2 u_\phi}{\partial t^2} \right)$$

$$\frac{\partial M_{\phi\theta}}{\partial \phi} + \frac{\partial M_{\theta\phi}}{\partial \theta} \operatorname{Cosec} \phi + (M_{\theta\phi} + M_{\phi\theta}) \cot \phi - a(V_\theta - m_\theta)$$

$$= \frac{1}{12} \rho h^3 a \left( k_r \frac{\partial^2 \beta_\theta}{\partial t^2} + \frac{c_r}{a} \frac{\partial^2 u_\theta}{\partial t^2} \right)$$

$$N_{\phi\theta} - N_{\theta\phi} = \frac{1}{a} (M_{\theta\phi} - M_{\phi\theta}) = 0^*$$

where

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\*The corresponding sixth equilibrium equation derived in [8] reads

$$N_{\phi\theta} - N_{\theta\phi} = \frac{1}{a} (M_{\theta\phi} - M_{\phi\theta}).$$



$$k_1 = 1 + \alpha, \quad k_2 = 2\alpha \quad (2.18)$$

$$k_r = 1 + \frac{9}{5} \alpha, \quad c_r = 2$$

and

$$\begin{bmatrix} p_\phi \\ p_\theta \\ p_n \end{bmatrix} = \frac{1}{4} \begin{bmatrix} p_\phi^+ H_\phi^+ - p_\phi^- H_\phi^- \\ p_\theta^+ H_\theta^+ - p_\theta^- H_\theta^- \\ p_n^+ H_n^+ - p_n^- H_n^- \end{bmatrix} \quad (2.19)$$

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## CHAPTER III

FORMULATION OF PROBLEMS FOR THE  
AXISYMMETRIC VIBRATION OF HEMISPHERICAL SHELLS

Governing Differential Equations

By setting  $k_r = k_2 = c_r = k_f = \lambda_f = 0$ , and  $k_1 = 1$  in equations (2.17) and taking the  $\theta$ -variation to be zero, one obtains the governing equations of torsionless motion for the axisymmetric deformation of a spherical shell without the effect from the transverse shear deformation. They are

$$\frac{\partial N_\phi}{\partial \phi} + (N_\phi - N_\theta) \cot \phi - V_\phi = \rho h a \frac{\partial^2 u_\phi}{\partial t^2} - a p_\phi \quad (3.1)$$

$$\frac{\partial V_\phi}{\partial \phi} + V_\phi \cot \phi + (N_\phi + N_\theta) = \rho h a \frac{\partial^2 w}{\partial t^2} - a p_n$$

$$\frac{\partial M_\phi}{\partial \phi} + (M_\phi - M_\theta) \cot \phi - a V_\phi = 0$$

Here the inward normal direction is considered to be positive. The stress, strain, and displacement relations are

$$N_\phi = \frac{Eh}{1-\nu^2} (\epsilon_\phi^\circ + \nu \epsilon_\theta^\circ) - \frac{E\alpha_0 \theta_0^*}{1-\nu} \quad (3.2)$$

$$\begin{aligned}
&= \frac{Eh}{a(1-\nu^2)} \left[ \left( \frac{\partial u}{\partial \phi} - w \right) + \nu(u_\phi \cot \phi - w) \right] - \frac{E\alpha_0}{1-\nu} \theta_0^* \\
N_\theta &= \frac{Eh}{1-\nu^2} (\epsilon_\theta^0 + \nu \epsilon_\phi^0) - \frac{E\alpha_0 \theta_0^*}{1-\nu} \\
&= \frac{Eh}{a(1-\nu^2)} \left[ (u_\phi \cot \phi - w) + \nu \left( \frac{\partial u}{\partial \phi} - w \right) \right] - \frac{E\alpha_0 \theta_0^*}{1-\nu} \\
M_\phi &= -D(K_\phi + \nu K_\theta) - \frac{E\alpha_0 \theta_1^*}{1-\nu} \\
&= -\frac{D}{a^2} \left[ \nu \left( u_\phi + \frac{\partial w}{\partial \phi} \right) \cot \phi + \left( \frac{\partial u}{\partial \phi} + \frac{\partial^2 w}{\partial \phi^2} \right) \right] - \frac{E\alpha_0 \theta_1^*}{1-\nu} \\
M_\theta &= -D(K_\theta + \nu K_\phi) - \frac{E\alpha_0 \theta_1^*}{1-\nu} \\
&= -\frac{D}{a^2} \left[ \left( u_\phi + \frac{\partial w}{\partial \phi} \right) \cot \phi + \nu \left( \frac{\partial u}{\partial \phi} + \frac{\partial^2 w}{\partial \phi^2} \right) \right] - \frac{E\alpha_0 \theta_1^*}{1-\nu}
\end{aligned}$$

where  $\alpha_0$ , the coefficient of linear thermal expansion, is assumed to be constant throughout the thickness of the shell and

$$\theta_0^* = \int_{-h/2}^{h/2} \theta^* d\zeta \quad (3.3)$$

$$\theta_1^* = \int_{-h/2}^{h/2} \theta^* \zeta d\zeta$$

with  $\theta^*$  denoting the temperature.

By eliminating  $V_\phi$  and substituting the stress-displacement relations in the first and second equations of (3.1), the following governing differential equations are obtained if  $\alpha$  is considered small compared to 1:

$$\alpha L(w) + L(\psi) - (1+\nu)w + (1-\nu)\psi - \lambda \frac{\partial^2 \psi}{\partial t^2} \quad (3.4)$$

$$= - \int_0^\phi \left[ A_1 p_\phi + B_1 Q_{T1} \right] d\phi$$

$$\alpha \left[ LL(w+\psi) + (1-\nu)L(w) \right] + (1+\nu)L(\psi) + 2(1+\nu)w + \lambda \frac{\partial^2 w}{\partial t^2}$$

$$= A_1 p_n + B_1 Q_{T2}$$

where

$$\alpha = \frac{1}{12} \left( \frac{h}{a} \right)^2$$

$$A_1 = \frac{a^2(1-\nu^2)}{Eh}, \quad B_1 = \frac{\alpha_0(1+\nu)}{h}$$

$$\lambda = \rho \frac{a^2(1-\nu^2)}{E}$$

$$u_{\phi} = \frac{\partial \psi}{\partial \phi}$$

$$Q_{T1} = - \left( a \frac{\partial \theta_0^*}{\partial \phi} - \frac{\partial \theta_1^*}{\partial \phi} \right)$$

$$Q_{T2} = - \left( 2a\theta_0^* + \frac{\partial \theta_1^*}{\partial \phi} \cot \phi + \frac{\partial^2 \theta_1^*}{\partial \phi^2} \right)$$

and

$$L(\cdot) = (1-x^2) \frac{\partial^2(\cdot)}{\partial x^2} - 2x \frac{\partial(\cdot)}{\partial x}$$

with

$$x = \cos \phi$$

Note that in equations (3.4),  $\theta_0^*$  and  $\theta_1^*$  may be functions of  $\phi$ .

#### Boundary and Initial Conditions

The boundary conditions corresponding to the edge supports shown in Figure 2 are

(I) Roller-clamped edge

$$\left. \begin{aligned} u_{\phi} = \frac{\partial \psi}{\partial \phi} &= 0 \\ \frac{\partial w}{\partial \phi} &= 0 \\ Q_{\phi} &= 0 \end{aligned} \right\} \text{ at } \phi = \frac{\pi}{2} \quad (3.5)$$

(II) Roller-hinged edge

$$\left. \begin{aligned} w &= 0 \\ N_{\phi} &= 0 \\ M_{\phi} &= 0 \end{aligned} \right\} \text{ at } \phi = \frac{\pi}{2} \quad (3.6)$$

(III) Clamped edge

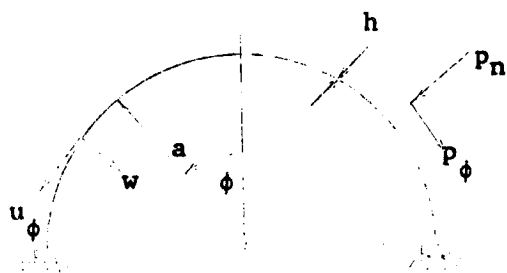
$$\left. \begin{aligned} u_{\phi} = \frac{\partial \psi}{\partial \phi} &= 0 \\ \frac{\partial w}{\partial \phi} &= 0 \\ w &= 0 \end{aligned} \right\} \text{ at } \phi = \frac{\pi}{2} \quad (3.7)$$

(IV) Hinged edge

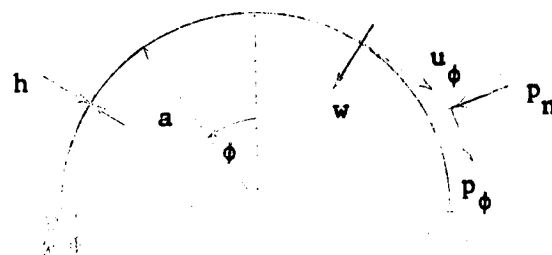
$$\left. \begin{aligned} u_{\phi} = \frac{\partial \psi}{\partial \phi} &= 0 \\ w &= 0 \\ M_{\phi} &= 0 \end{aligned} \right\} \text{ at } \phi = \frac{\pi}{2} \quad (3.8)$$

The initial conditions which will be considered are

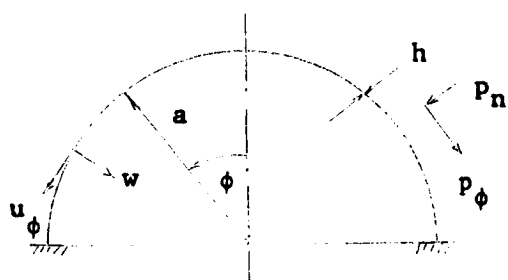
$$\left. \begin{aligned} u_{\phi} = w &= 0 \\ \frac{\partial u}{\partial t} = \frac{\partial w}{\partial t} &= 0 \end{aligned} \right\} \text{ at } t = 0 \quad (3.9)$$



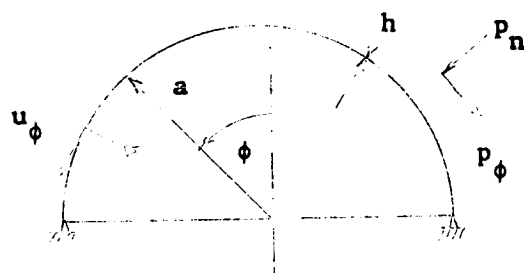
Roller-Clamped Edge



Roller-Hinged Edge



Clamped Edge



Hinged Edge

Figure 2. Geometry and the Boundary Conditions



The general solutions of equations (3.4) governing the displacements of spherical shells subjected to surface loading and/or thermal shock will include the following three parts:

- (A) Homogeneous solution.
- (B) Particular solution corresponding to surface loading alone.
- (C) Particular solution corresponding to thermal effect alone.

The final solutions will be obtained by satisfying the appropriate boundary conditions shown in equations (3.5) through (3.8).

It should be noted that when the thermal effect is considered, the homogeneous stress boundary conditions expressed in terms of displacements will become inhomogeneous boundary conditions in the displacements.

## CHAPTER IV

## DYNAMIC AND THERMAL RESPONSE

By using the modal analysis, closed form solutions are obtained for the axisymmetric dynamic response of hemispherical shells with roller-hinged edges and roller-clamped edges. By applying a harmonic ring load to withhold the motion either along the longitudinal direction of the roller-hinged edge or along the transverse direction of the roller-clamped edge, the eigenfunctions for the hinged and clamped edges are obtained. In solving the problem including thermal loading, the inhomogeneous boundary conditions in the displacements arising from the thermal effect must be considered. A particular term is added to the formal solution of the problem with homogeneous boundary conditions in the displacements such that the resulting solution now satisfies the governing differential equations and the boundary conditions.

The governing differential equations are those of equations (3.4). They are

$$L(\psi) = -\alpha L(w) + (1+\nu) w - (1-\nu)\psi \quad (4.1)$$

$$+ \lambda \frac{\partial^2 \psi}{\partial t^2} - A_1 \int_0^\phi p_\phi d\phi - B_1 \int_0^\phi Q_{T1} d\phi$$

$$LL(w) = -LL(\psi) - (1-\nu) L(w) + \frac{1+\nu}{\alpha} L(\psi) \quad (4.2)$$

$$-\frac{2(1+\nu)}{\alpha} w - \frac{\lambda}{\alpha} \frac{\partial^2 w}{\partial t^2} + p_n \frac{A_1}{\alpha} + Q_{T2} \frac{B_1}{\alpha}$$

Dynamic Response of Shells Having Roller-Clamped  
and Roller-Hinged Supports

When the effect of the temperature  $\theta^*$  is neglected the boundary conditions (3.5) and (3.6) at  $\phi = \frac{\pi}{2}$  are

(I) Roller-Clamped Edge

$$\frac{\partial \psi}{\partial \phi} = 0 \quad (4.3)$$

$$\frac{\partial w}{\partial \phi} = 0 \quad (4.4)$$

$$\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^3 w}{\partial \phi^3} = 0 \quad (4.5)$$

(II) Roller-Hinged Edge

$$w = 0 \quad (4.6)$$

$$\frac{\partial u}{\partial \phi} = 0 \quad (4.7)$$

$$\frac{\partial^2 w}{\partial \phi^2} = 0 \quad (4.8)$$

The solutions of equations (4.1) and (4.2) which satisfy the boundary conditions (4.3) to (4.5) and (4.6) to (4.8) corresponding to the roller-clamped and roller-hinged edges, respectively, can be expressed in terms of the Legendre's polynomials in the following forms:

$$\psi(x,t) = \sum_{\substack{n=0,2,4,\dots \\ \text{or} \\ n=1,3,5,\dots}}^{\infty} \psi_n(t) P_n(x) \quad (4.9)$$

$$w(x,t) = \sum_{\substack{n=0,2,4,\dots \\ \text{or} \\ n=1,3,5,\dots}}^{\infty} w_n(t) P_n(x) \quad (4.10)$$

where  $n = 0, 2, 4, \dots$  corresponds to the roller-clamped case, and  $n = 1, 3, 5, \dots$  corresponds to the roller-hinged case. From the physical viewpoint,  $\psi$  and  $w$  must be continuous functions of  $x$  and possess bounded first derivatives. Therefore the series given in equations (4.9) and (4.10) will converge absolutely and uniformly.\*

Substitution of equations (4.9) and (4.10) into equations (4.1) and (4.2) yields

$$\sum_{\substack{n=0,2,4,\dots \\ \text{or} \\ n=1,3,5,\dots}}^{\infty} \left\{ \psi_n(t) \left[ L(P_n) + (1-\nu)P_n \right] - \lambda \frac{d^2 \psi_n}{dt^2} P_n \right. \quad (4.11) \\ \left. + w_n(t) \left[ L(P_n) - (1+\nu)P_n \right] \right\} = -A_1 \int_0^\phi p_\phi d\phi$$

---

\*See Reference 1, pp. 424-429.

$$\begin{aligned}
& \sum_{\substack{n=0,2,4,\dots \\ \text{or} \\ n=1,3,5,\dots}}^{\infty} \left\{ \psi_n(t) \left[ LL(P_n) - \frac{(1+\nu)}{\alpha} L(P_n) \right] + w_n(t) \left[ LL(P_n) \right. \right. \\
& \quad \left. \left. + (1-\nu)L(P_n) + \frac{2(1+\nu)}{\alpha} P_n \right] + \frac{\lambda}{\alpha} \frac{d^2 w_n}{dt^2} P_n \right\} = p_n \frac{A_1}{\alpha} \quad (4.12)
\end{aligned}$$

By using the Legendre's identity

$$L(P_n) = -\Lambda_n P_n \quad (4.13)$$

with

$$\Lambda_n = n(n+1)$$

and considering orthogonality condition, equations (4.11) and

(4.12) become

$$\begin{aligned}
\lambda \frac{d^2 \psi_n}{dt^2} &= \left[ (1-\nu) - \Lambda_n \right] \psi_n + \left[ \alpha \Lambda_n + (1+\nu) \right] w_n \\
&= A_1 (2n+1) \int_0^1 P_n(\xi) \int_0^{\phi(\xi)} p_\phi d\phi d\xi \quad (4.14)
\end{aligned}$$

$$\begin{aligned}
\lambda \frac{d^2 w_n}{dt^2} &+ \left\{ 2(1+\nu) + \alpha \Lambda_n \left[ \Lambda_n - (1-\nu) \right] \right\} w_n \\
&+ \Lambda_n \left[ (1+\nu) + \alpha \Lambda_n \right] \psi_n = A_1 (2n+1) \int_0^1 p_n P_n(\xi) d\xi \quad (4.15)
\end{aligned}$$

According to the definitions after equations (3.4) and using equation

(4.9), the relations

$$u_\phi(x, t) = \frac{\partial \psi(x, t)}{\partial x} \frac{dx}{d\phi} = \sum_{n=0,2,4,\dots}^{\infty} \psi_n(t) \frac{dP_n(x)}{dx} \frac{dx}{d\phi} \quad (4.16)$$

yield

$$u_{\phi}(x, t) = \sum_{n=2,4,\dots}^{\infty} \psi_n(t) \frac{dP_n(x)}{dx} \frac{dx}{d\phi} \quad (4.17)$$

$\psi_0$  has no contribution to the response  $u_{\phi}$  and is discarded from the analysis. Letting  $n = 0$ , equation (4.15) becomes

$$\lambda \frac{d^2 w_0}{dt^2} + 2(1+\nu)w_0 = A_1 \int_0^1 P_n(\xi, \tau) d\xi \quad (4.18)$$

Equation (4.18) has a solution

$$w_0 = \frac{A_1}{\lambda \omega_{10}} \int_0^t \sin \omega_{10}(t-\tau) \int_0^1 p_n(\xi, \tau) d\xi d\tau \quad (4.19)$$

where

$$\omega_{10} = \left[ \frac{2(1+\nu)}{\lambda} \right]^{1/2} \quad (4.20)$$

represents the breathing-mode\* frequency of the roller-clamped shell.

For  $n = 1$  equations (4.14) and (4.15) become

$$\begin{aligned} \lambda \frac{d^2 \psi}{dt^2} + (1+\nu)\psi_1 + [2\alpha + (1+\nu)] w_1 \\ = 3A_1 \int_0^1 P_1(\xi) \int_0^{\phi(\xi)} p_{\phi} d\phi d\xi \end{aligned} \quad (4.21)$$

$$\begin{aligned} \lambda \frac{d^2 w_1}{dt^2} + 2(1+\nu)(1+\alpha)w_1 + 2 [2\alpha + (1+\nu)] \psi_1 \\ = 3A_1 \int_0^1 p_n P_1(\xi) d\xi \end{aligned}$$

---

\*Mode shape is described by a constant normal displacement  $w$  over the entire shell surface.

If  $\alpha$  is small compared to 1 and equations (4.21) are rearranged, one obtains

$$\lambda \frac{d^2(w_1 - 2\psi_1)}{dt^2} = 3A_1 \left[ \int_0^1 p_n p_1(\xi) d\xi - 2 \int_0^1 p_1(\xi) \int_0^{\phi(\xi)} p_\phi d\phi d\xi \right] \quad (4.22)$$

$$\begin{aligned} \lambda \frac{d^2(w_1 + \psi_1)}{dt^2} + 3(1+\nu)(w_1 + \psi_1) \\ = 3A_1 \left[ \int_0^1 p_n p_1(\xi) d\xi + \int_0^1 p_1(\xi) \int_0^{\phi(\xi)} p_\phi d\phi d\xi \right] \end{aligned}$$

The first of equations (4.22) can be integrated directly and the solutions of equations (4.22) are

$$w_1 - 2\psi_1 = \frac{1}{\lambda} \int_0^t \int_0^\tau F_{11}(\tau') d\tau' d\tau \quad (4.23)$$

$$w_1 + \psi_1 = \frac{1}{\lambda \omega_{11}} \int_0^t F_{21}(\tau) \sin \omega_{11}(t-\tau) d\tau$$

where

$$\omega_{11} = \sqrt{\frac{3(1+\nu)}{\lambda}} \quad (4.24)$$

is the 1<sup>st</sup> natural frequency\* for the roller-hinged case, and

---

\*The equations for computing the natural frequencies are those of (4.30).

$$\left. \begin{aligned} F_{11} &= 3A_1 \int_0^1 p_n(\xi, \tau) P_1(\xi) d\xi - 2 \int_0^1 P_1(\xi) \int_0^{\phi(\xi)} p_\phi(\phi, \tau) d\phi d\xi \\ F_{21} &= 3A_1 \int_0^1 p_n(\xi, \tau) P_1(\xi) d\xi + 2 \int_0^1 P_1(\xi) \int_0^{\phi(\xi)} p_\phi(\phi, \tau) d\phi d\xi \end{aligned} \right\} \quad (4.25)$$

$\psi_1$  and  $w_1$  are then

$$\begin{aligned} \psi_1 &= \frac{1}{3\lambda} \left[ \frac{1}{\omega_{11}} \int_0^t F_{21}(\tau) \sin \omega_{11}(t-\tau) d\tau \right. \\ &\quad \left. - \int_0^t \int_0^\tau F_{11}(\tau') d\tau' d\tau \right] \\ w_1 &= \frac{1}{3\lambda} \left[ \frac{1}{\omega_{11}} \int_0^t F_{21}(\tau) \sin \omega_{11}(t-\tau) d\tau \right. \\ &\quad \left. + \int_0^t \int_0^\tau F_{11}(\tau') d\tau' d\tau \right] \end{aligned} \quad (4.26)$$

Finally for  $n \geq 2$  the equations of (4.14) and (4.15) are rearranged to yield two uncoupled fourth-order ordinary differential equations.

They are

$$\lambda^2 \frac{d^4 \psi_n}{dt^4} + A_{1n} \frac{d^2 \psi_n}{dt^2} + A_{2n} \psi_n = F_{1n}(t) \quad (4.27)$$

$$\lambda^2 \frac{d^4 w_n}{dt^4} + A_{1n} \frac{d^2 w_n}{dt^2} + A_{2n} w_n = F_{2n}(t)$$



where

$$A_{1n} = \alpha \Lambda_n (\Lambda_n - (1-\nu)) + \Lambda_n + (1+3\nu) \quad (4.28)$$

$$A_{2n} = \alpha \left[ \Lambda_n (\Lambda_n - 1 + \nu)^2 - \alpha^2 \Lambda_n^3 - 2\alpha \Lambda_n^3 (1+\nu) \right] + (1-\nu^2) (\Lambda_n - 2)$$

$$F_{1n}(t) = A_1(2n+1) \left[ A_{3n} + \lambda \frac{d^2}{dt^2} \right] \int_0^1 P_n(\xi) \int_0^{\phi(\xi)} p_\phi(\phi, t) d\phi d\xi \\ - \left[ \alpha \Lambda_n + (1+\nu) \right] \int_0^1 p_\phi(\xi, t) P_n(\xi) d\xi$$

$$F_{2n}(t) = A_1(2n+1) \left[ \Lambda_n - (1-\nu) + \lambda \frac{d^2}{dt^2} \right. \\ \cdot \left. \int_0^1 p_n(\xi, t) P_n(\xi) d\xi + \Lambda_n \left[ \alpha \Lambda_n + (1+\nu) \right] \right. \\ \cdot \left. \int_0^1 P_n(\xi) \int_0^{\phi(\xi)} p_\phi(\phi, t) d\phi d\xi \right]$$

and

$$A_{3n} = \alpha \Lambda_n (\Lambda_n - 1 + \nu) + 2(1 + \nu)$$

Solutions of equations (4.27) may be written in terms of convolution integrals as

$$\psi_n = \frac{1}{\lambda^2(\omega_{2n}^2 - \omega_{1n}^2)} - \frac{1}{\omega_{11}} \int_0^t \sin \omega_{1n}(t-\tau) F_{1n}(\tau) d\tau \quad (4.29)$$

$$- \frac{1}{\omega_{2n}} \int_0^t \sin \omega_{2n}(t-\tau) F_{1n}(\tau) d\tau$$

$$w_n = \frac{1}{\lambda^2(\omega_{2n}^2 - \omega_{1n}^2)} - \frac{1}{\omega_{11}} \int_0^t \sin \omega_{1n}(t-\tau) F_{2n}(\tau) d\tau$$

$$- \frac{1}{\omega_{2n}} \int_0^t \sin \omega_{2n}(t-\tau) F_{2n}(\tau) d\tau$$

where

$$\omega_{1n}^2 = \frac{1}{2\lambda} \left[ A_{1n} + (A_{1n}^2 - 4A_{2n})^{1/2} \right] \quad (4.30)$$

$$\omega_{2n}^2 = \frac{1}{2\lambda} \left[ A_{1n} - (A_{1n}^2 - 4A_{2n})^{1/2} \right]$$

represent the upper and lower branches of the natural frequencies, respectively.

$\lambda \omega_{1n}^2$  and  $\lambda \omega_{2n}^2$  are plotted for  $\frac{h}{a} = 0.01, 0.02$ , and  $0.05$  and are shown in Figure 3. It is seen that the variation of thickness has very little effect on the upper branch frequencies. However, the lower branch frequencies change significantly as the thickness of the shell increases, particularly at the higher modes. In addition, the magnitudes of the lower branch frequencies are not bounded for the higher modes. This contradicts the results obtained by membrane theory.

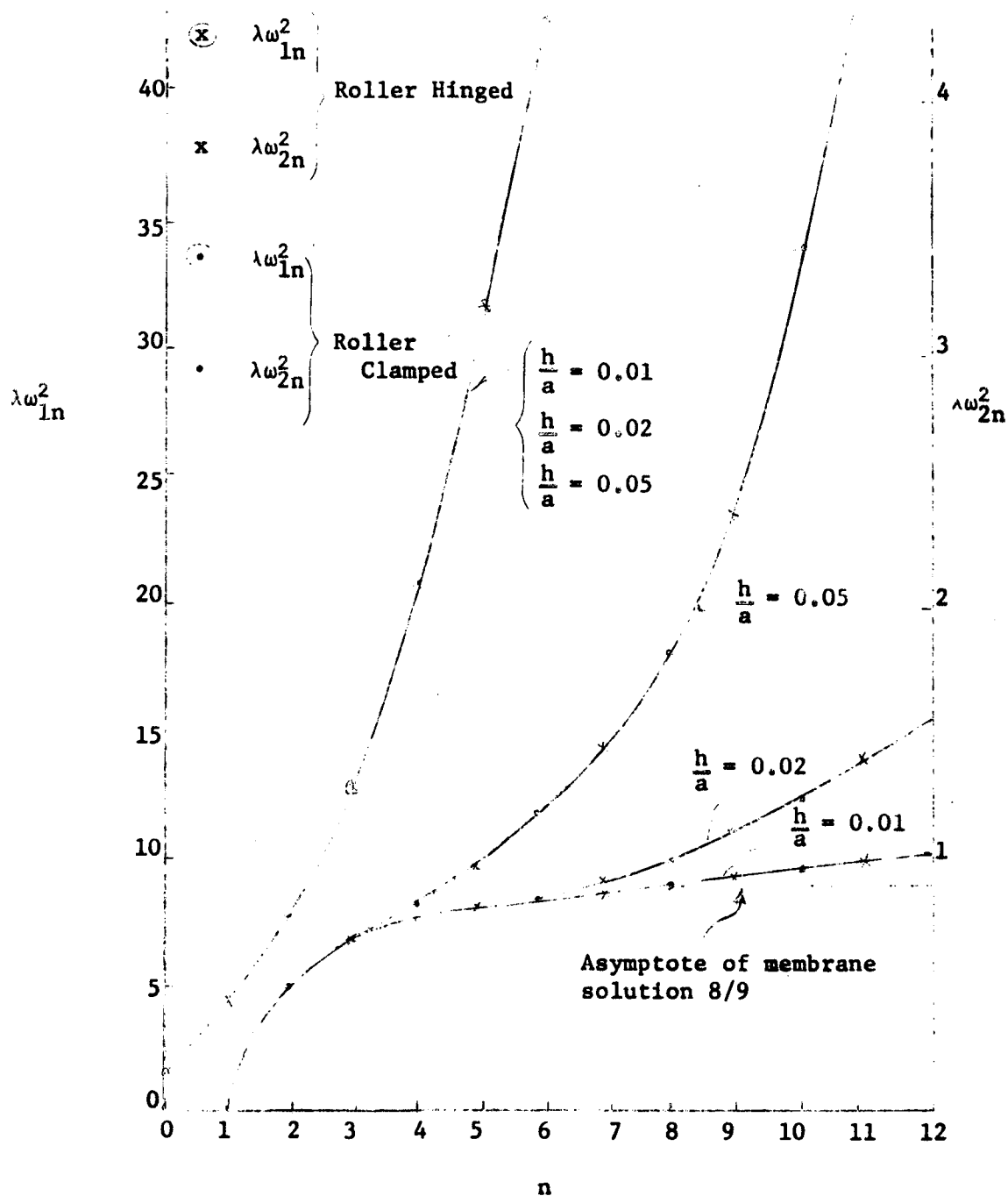


Figure 3. Natural Frequencies of Hemispherical Shells with Roller-Hinged and Roller-Clamped Boundary Conditions,  $\nu = \frac{1}{3}$

The final solutions for the displacements are

$$u_{\phi} = - \sum_{\substack{n=2,4,\dots \\ \text{or} \\ n=1,3,5,\dots}}^{\infty} \psi_n(t) \frac{dP_n(x)}{dx} \quad 1-x^2 \quad (4.31)$$

$$w = \sum_{\substack{n=0,2,4,\dots \\ \text{or} \\ n=1,3,5,\dots}}^{\infty} w_n(t) P_n(x)$$

#### Free Vibration of Clamped Shells

The boundary conditions for the present case are

$$\begin{aligned} u_{\phi} = \frac{\partial \psi}{\partial \phi} &= 0 \\ \frac{\partial w}{\partial \phi} &= 0 \quad \text{at } \phi = \frac{\pi}{2} \\ w &= 0 \end{aligned} \quad (4.32)$$

The solution for the free vibration of the shell may be obtained by consideration of the following harmonic edge load acting on a shell having a roller-clamped support:

$$\begin{aligned} p_{\phi} &= 0 \\ p_n &= p_0 \delta(x=0) e^{i\Omega t} \end{aligned} \quad (4.33)$$

Substituting equations (4.33) into equations (4.19) and (4.27)

with  $n$  being even integers, the steady state solution becomes

$$w = p_0 \frac{\lambda \Omega^2 + (1-\nu)}{\lambda (\Omega^2 - \omega_{10}^2)} + \sum_{n=2,4,6,\dots}^{\infty} \frac{(2n+1) \lambda \Omega^2 - \Lambda_n + (1-\nu)}{\lambda^2 (\Omega^2 - \omega_{1n}^2) (\Omega^2 - \omega_{2n}^2)} \quad (4.34)$$

$$= P_r(0) P_n(x) e^{i\Omega t}$$

The last condition given in equations (4.32) is applied to equation (4.34) and results in the following frequency equation:

$$S_h = \left. \frac{\lambda \Omega^2 + (1-\nu)}{\lambda (\Omega^2 - \omega_{10}^2)} + \sum_{n=2,4,\dots}^{\infty} \frac{(2n+1) [\lambda \Omega^2 - \Lambda_n + (1-\nu)]}{\lambda^2 (\Omega^2 - \omega_{1n}^2) (\Omega^2 - \omega_{2n}^2)} P_n^2(0) \right\} = 0 \quad (4.35)$$

The natural frequencies  $\Omega$  are obtained by using an iterative procedure on a Burrough 220 electronic computer. The roots of equation (4.35) are searched for between every two consecutive  $\lambda \omega_{in}^2$  ( $i=1$  and  $2$  correspond to upper and lower branches, respectively). A number of trial values of  $\lambda \Omega^2$  with constant increments are fed into equation (4.35). The curves  $S_h$  vs.  $\lambda \omega^2$  are plotted. The frequencies may be obtained by interpolating between the two consecutive values of  $\lambda \Omega^2$  where the corresponding values change signs. The value of  $n$  for the series solutions is then increased until the variation of frequencies obtained is acceptable. After the approximate values of  $\lambda \Omega^2$  are obtained, a more sophisticated numerical device similar to the method of false position may be programmed and used to search for more accurate  $\lambda \Omega^2$ 's in the vicinities of the approximate ones.

The first three natural frequencies for a hemispherical shell completely clamped along its edge with  $\frac{h}{a} = 0.01, 0.02, \text{ and } 0.05$ , and  $n = 60$ , obtained by interpolation, are listed in Table 1. The curves  $S_h$  vs.  $\lambda\omega^2$  are plotted in Figures 4 - 6, for various values of  $h/a$ . For the case  $\frac{h}{a} = 0.01$ , the largest difference between the three lowest frequencies for the roller-clamped and completely clamped is eight percent. The first three mode shapes for a clamped shell are plotted in Figures (7-a), (7-b), and (7-c).

Table 1.  $\lambda\Omega^2$  for Completely Clamped Shells with  $\nu = \frac{1}{3}$

	$\frac{h}{a} = 0.01$	$\frac{h}{a} = 0.02$	$\frac{h}{a} = 0.05$
1 <sup>st</sup> Mode	0.512	0.522	0.565
2 <sup>nd</sup> Mode	0.787	0.805	0.955
3 <sup>rd</sup> Mode	0.879	0.940	1.462

By applying the same technique on shells with roller-hinged edges, the natural frequencies of shells with hinged edges may be obtained.

#### Dynamic Response - Mode-Acceleration Solution

We introduce, for the present study, a method developed by Williams [2]. In this method the governing differential equations (4.1) and (4.2) are first written in the following vector form:

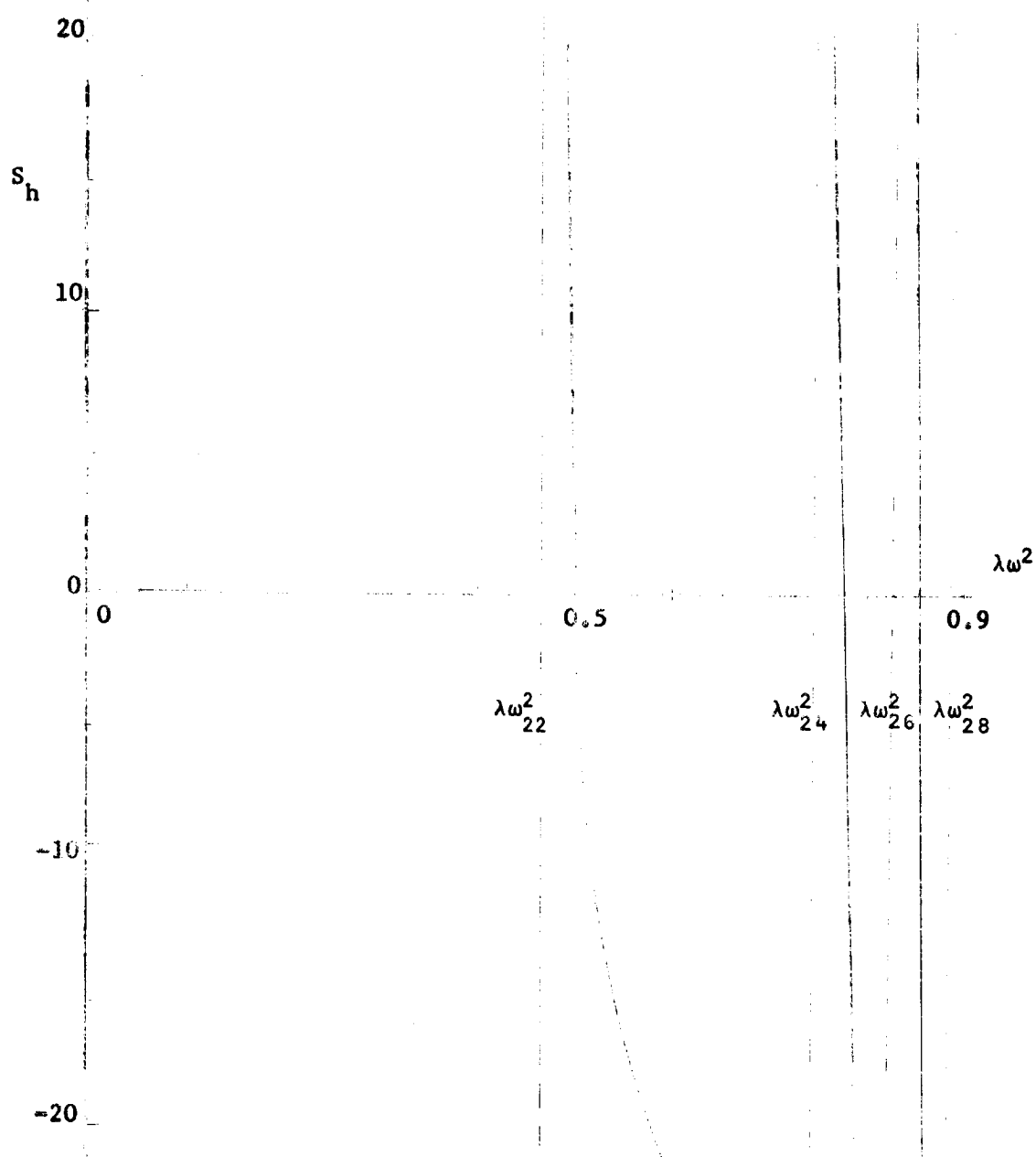


Figure 4.  $S_h$  vs.  $\lambda\omega^2$  for  $\frac{h}{a} = 0.01$  and  $\nu = \frac{1}{3}$

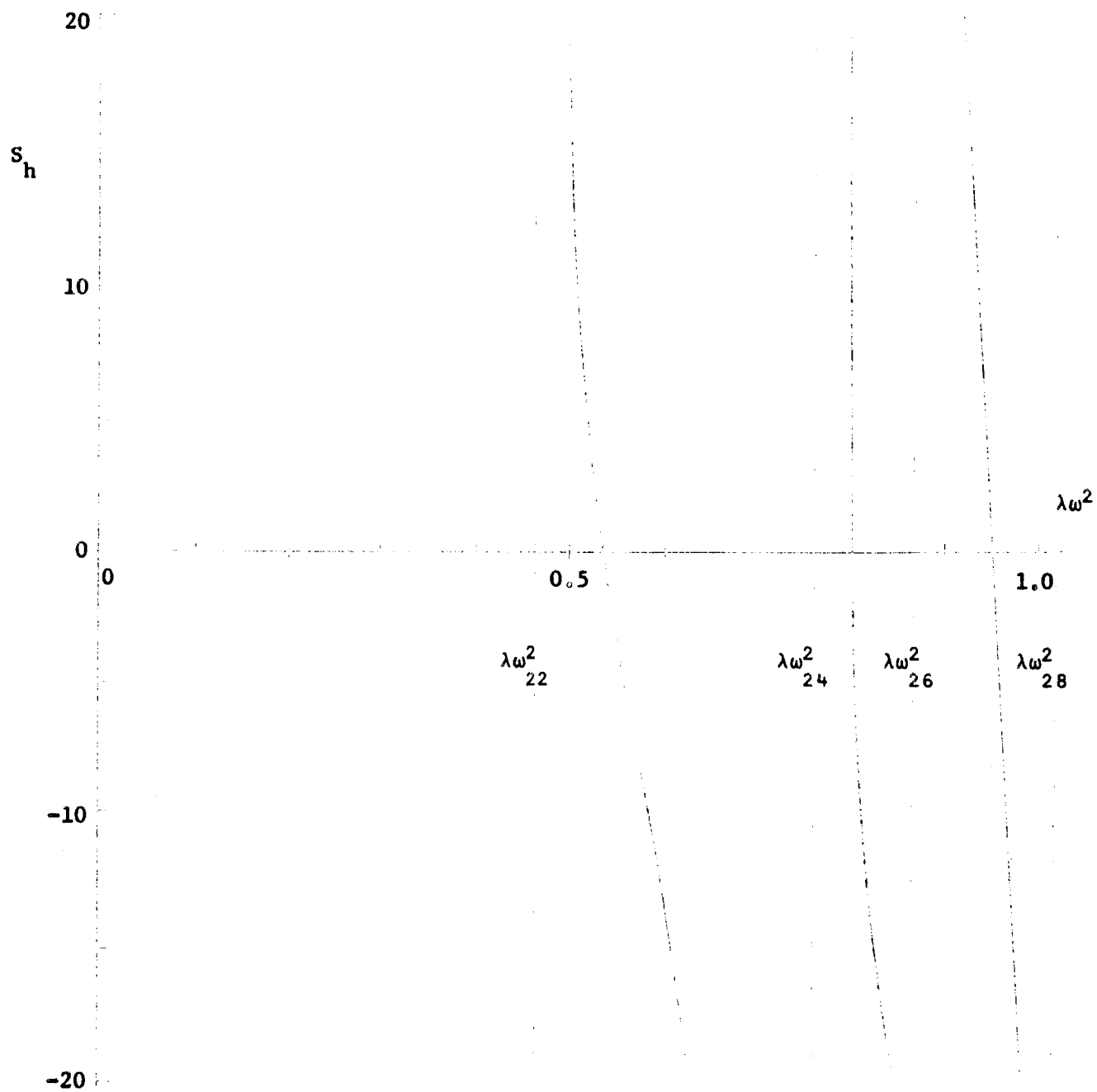


Figure 5.  $S_h$  vs.  $\lambda\omega^2$  for  $\frac{h}{a} = 0.02$  and  $\nu = \frac{1}{3}$



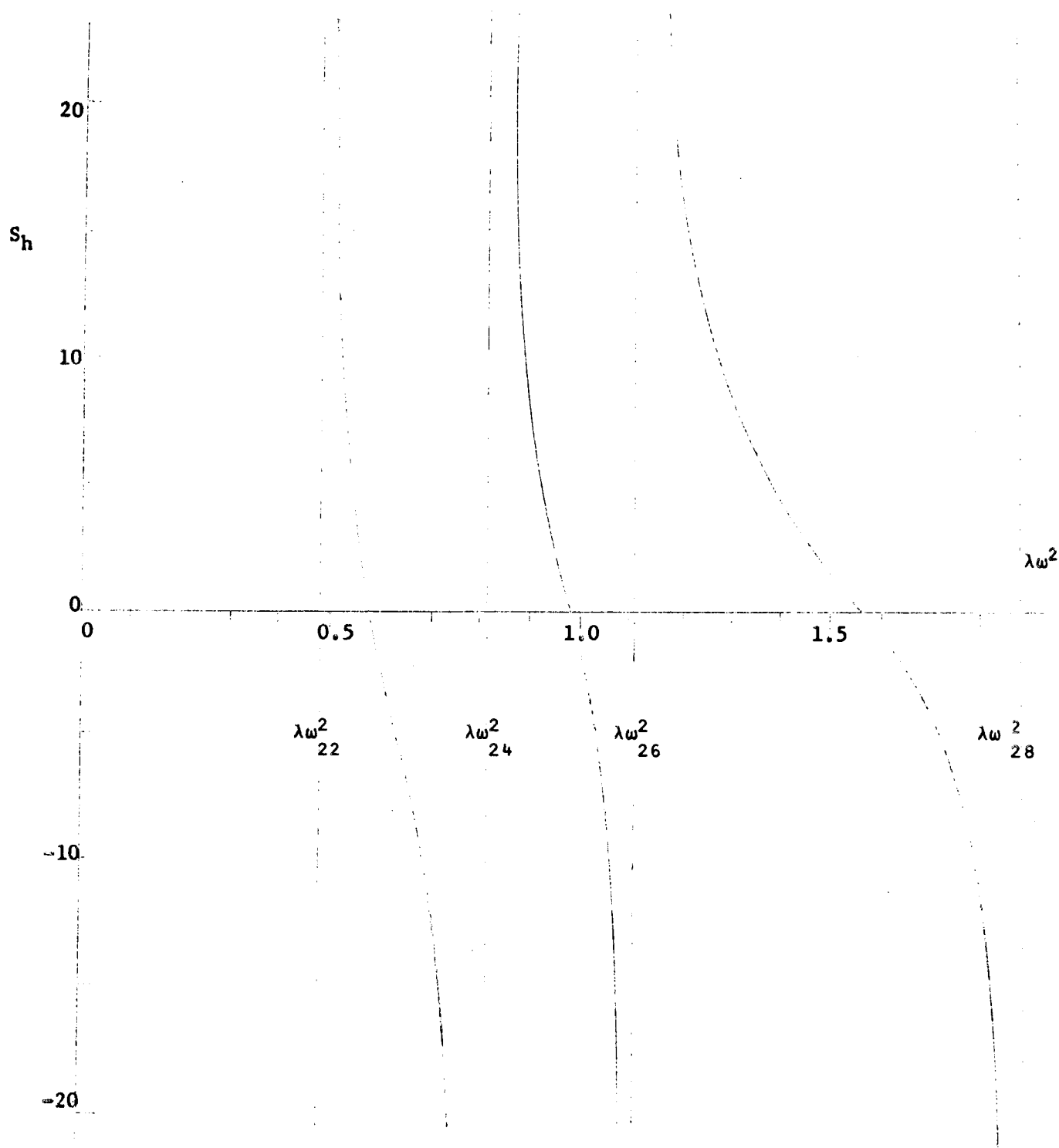


Figure 6.  $S_h$  vs.  $\lambda\omega^2$  for  $\frac{h}{a} = 0.05$  and  $\nu = \frac{1}{3}$

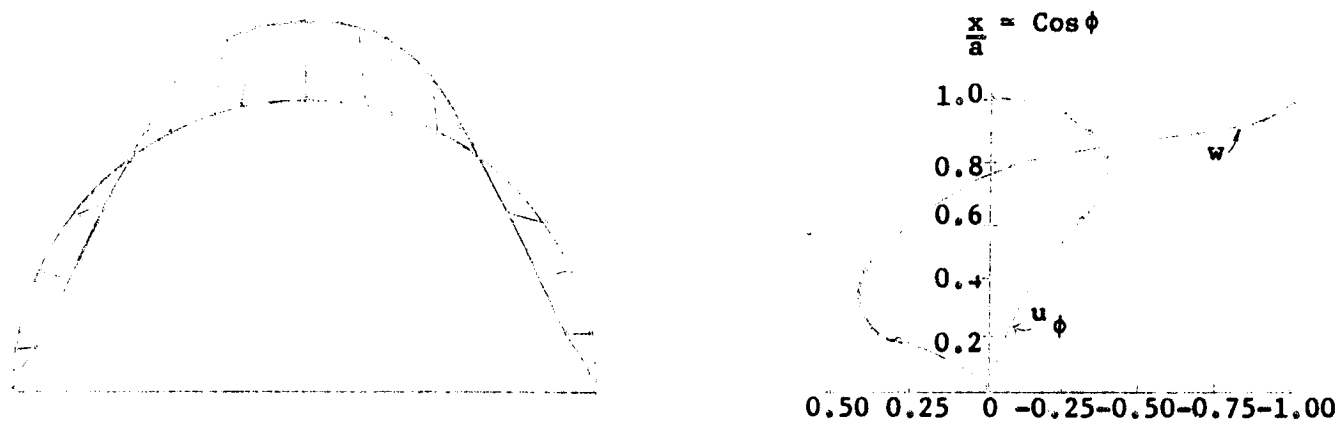


Figure (7-a). First Mode Shape of Hemispherical Shells with Clamped Boundary Conditions

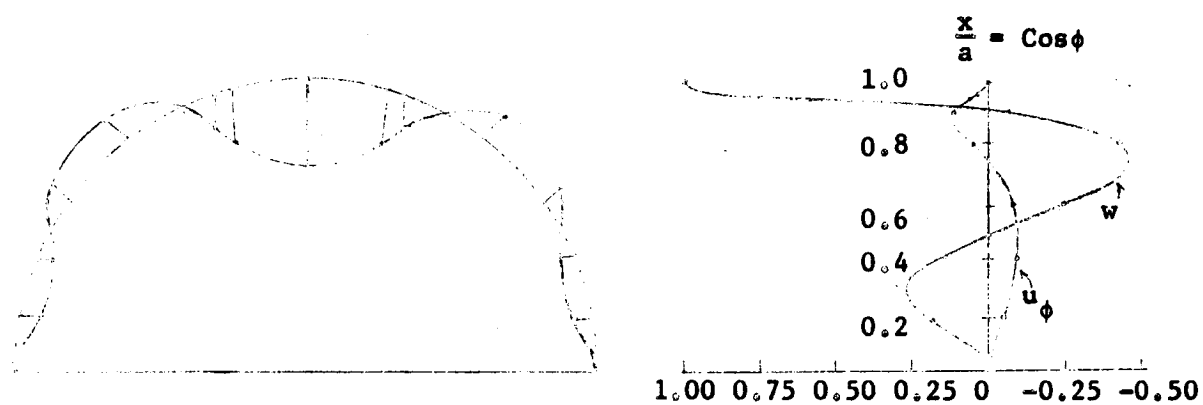
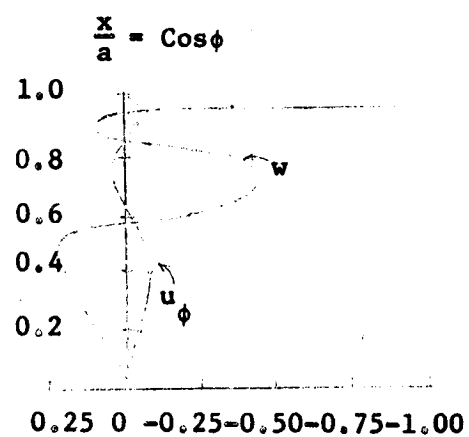
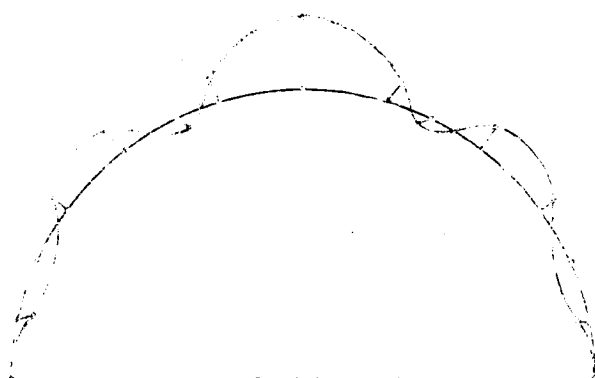


Figure (7-b). Second Mode Shape of Hemispherical Shells with Clamped Boundary Conditions



**Figure (7-c). Third Mode Shape of Hemispherical Shells with Clamped Boundary Conditions**

$$[L_{\alpha\beta}] \{u_\beta\} - \rho h \frac{\partial^2 u_\alpha}{\partial t^2} = \{p_\alpha\} \quad (4.36)$$

where  $[L_{\alpha\beta}]$  is a matrix differential operator, and where  $\{u_\alpha\}$  and  $\{p_\alpha\}$  are the displacement and load column vectors, respectively.

Assume the displacement vector, according to Williams' method [3,4], to be of the form

$$\{u_\alpha(x,t)\} = \{\bar{u}_\alpha(x,t)\} + \sum_j \phi_j(t) \{u_{\alpha j}(x)\} \quad (4.37)$$

where  $\{\bar{u}_\alpha\}$  and  $\{u_{\alpha j}\}$  are solutions of

$$[L_{\alpha\beta}] \{\bar{u}_\beta\} = \{p_\alpha\} \quad (4.38)$$

and

$$[L_{\alpha\beta}] \{u_{\beta j}\} + \rho h \Omega_j^2 \{u_{\alpha j}\} = 0 \quad (4.39)$$

respectively. Here, the  $\bar{u}_\alpha$  represent the quasi-static response; and the  $u_{\alpha j}$  are the normal modes of free vibration with corresponding eigenfrequencies  $\Omega_j$ . The necessary coupling equations for the evaluation of the  $\phi_j$  are obtained by substituting equation (4.37) into equation (4.36) and simplifying by means of equations (4.38) and (4.39). The resulting coupled equation is

$$\sum_{j=1}^{\infty} (\phi_j + \Omega_j^2 \phi_j) \{u_{\alpha j}\} = -\{\ddot{\bar{u}}_\alpha\} \quad (4.40)$$

Multiplication of both sides of equation (4.40) by the row vector  $[u_{\alpha i}]$  and use of the orthogonality condition, i.e.,

$$\int_S \sum_{\alpha=1}^3 u_{\alpha i} u_{\alpha j} dS = \delta_{ij} \int_S \sum_{\alpha=1}^3 u_{\alpha i}^2 dS \quad (4.41)$$

results in the following expression:

$$\ddot{\phi}_j + \Omega_j^2 \phi_j = - \frac{\int_S \sum_{\alpha=1}^3 u_{\alpha i} \ddot{u}_{\alpha} dS}{\int_S \sum_{\alpha=1}^3 u_{\alpha j}^2 dS} \quad (4.42)$$

where  $S$  is the middle surface of the shell.

When the quasi-static solution can be obtained exactly, this will speed up the convergence of the total solution. In general, however, the quasi-static solution simply represents a quasi-equilibrium position about which the dynamic response of the shell is distributed.

#### Thermally Induced Vibration

The governing differential equations are obtained from those of (3.4) by setting  $p_\phi = p_n = 0$ . They are

$$L(\psi) = -\alpha L(w) + (1+\nu)w - (1-\nu)\psi \quad (4.43)$$

$$+ \lambda \frac{\partial^2 \psi}{\partial t^2} - B_1 \int_0^\phi Q_{T1} d\phi$$

$$LL(w) = - LL(\psi) - (1-\nu)L(w) + \frac{(1+\nu)}{\alpha} L(\psi)$$

$$- \frac{2(1+\nu)}{\alpha} w - \frac{\lambda}{\alpha} \frac{\partial^2 w}{\partial t^2} + B_1 \frac{Q_{T2}}{\alpha}$$

Analysis of the present case is similar to the dynamic loading case except that the homogeneous stress boundary conditions expressed in terms of displacements will become inhomogeneous boundary conditions in the displacements.

Assume solutions for  $\psi$  and  $w$  of the following types:

$$\psi = \sum_{n=1,3,5,\dots}^{\infty} \psi_n(t) P_n(x) + B_2 G_{1\psi}(x,t) \quad (4.44)$$

or  
 $n=0,2,4,\dots$

$$w = \sum_{n=1,3,5,\dots}^{\infty} w_n(t) P_n(x) + B_2 G_{1w}(x,t)$$

or  
 $n=0,2,4,\dots$

where

$$B_2 = \frac{\alpha_0(1+\nu)}{h} \quad (4.45)$$

$G_{1\psi}(x,t)$  and  $G_{1w}(x,t)$  are functions to be

determined by the boundary conditions with  $i = 1, 2$  representing the roller-hinged and roller-clamped boundary cases, respectively.

Substituting equations (4.44) into equations (4.43) and then integrating over  $[0, 1]$ , two coupled ordinary differential equations similar to equations (4.14) and (4.15) are obtained.

For  $n = 0$ ,

$$\lambda \frac{d^2 w_0}{dt^2} + 2(1+\nu)w_0 = B_2 \int_0^1 \{Q_{T2} + H_{iw_0}\} d\xi \quad (4.46)$$

where

$$H_{iw_0} = \left[ -\alpha LL - \alpha(1-\nu)L - 2(1+\nu) - \lambda \frac{\partial^2}{\partial t^2} \right] G_{iw} \quad (4.47)$$

$$+ [\alpha LL + (1+\nu)] G_{i\psi} \Big|_{n=0}$$

Equation (4.46) has a solution

$$w_0 = \frac{B_2}{\lambda \omega_{10}} \int_0^t \sin \omega_{10}(t-\tau) \int_0^1 \{Q_{T2} + H_{iw_0}\} d\xi d\tau \quad (4.48)$$

The solutions for  $\psi$  and  $w$  for  $n \geq 1$  are obtained by similar substitutions and manipulations. They are

$$\psi_1 = \frac{1}{3\lambda} \left\{ \frac{1}{\omega_{11}} \int_0^t F_{1w}(\tau) \sin \omega_{11}(t-\tau) d\tau \right. \quad (4.49)$$

$$\left. - \int_0^t \int_0^\tau F_{1\psi}(\tau') d\tau' d\tau \right\}$$



$$w_1 = \frac{1}{3\lambda} \left\{ \frac{2}{\omega_{11}} \int_0^t F_{1w}(\tau) \sin \omega_{11}(t-\tau) d\tau \right. \\ \left. + \int_0^t \int_0^\tau F_{1\psi}(\tau') d\tau' d\tau \right\}$$

with

$$F_{1\psi} = 3B_2 \left\{ \int_0^1 [Q_{T2} + H_{iw_1}] P_1(\xi) d\xi \right. \\ \left. - 2 \int_0^1 \left[ \int_0^{\phi(\xi)} Q_{T1} d\phi d\xi + H_{i\psi_1} \right] P_1(\xi) d\xi \right\} \quad (4.50a)$$

$$F_{1w} = 3B_2 \left\{ \int_0^1 [Q_{T2} + H_{iw_1}] P_1(\xi) d\xi \right. \\ \left. + \int_0^1 \left[ \int_0^{\phi(\xi)} Q_{T1} d\phi + H_{i\psi_1} \right] P_1(\xi) d\xi \right\}$$

and

$$H_{i\psi_1} = \frac{1}{3} \left\{ \left[ L + (1-\nu) - \lambda \frac{\partial^2}{\partial t^2} \right] G_{i\psi} \right. \\ \left. + [\alpha L - (1+\nu)] G_{iw} \right\} \Big|_{n=1} \quad (4.50b)$$

$$H_{iw_1} = \frac{1}{3} \left\{ \left[ -\alpha LL - (1-\nu)\alpha L - 2(1+\nu) - \lambda \frac{\partial^2}{\partial t^2} \right] G_{iw} \right.$$

$$+ [-\alpha L L + (1+\nu)L] G_{i\psi} \Bigg\} \Bigg|_{n=1}$$

For  $n \geq 2$ ,

$$\begin{aligned} \psi_n &= \frac{1}{\lambda^2(\omega_{2n}^2 - \omega_{1n}^2)} \left\{ \frac{1}{\omega_{1n}} \int_0^t \text{Sin} \omega_{1n}(t-\tau) F_{n\psi}(\tau) d\tau \right. \\ &\quad \left. - \frac{1}{\omega_{2n}} \int_0^t \text{Sin} \omega_{2n}(t-\tau) F_{n\psi}(\tau) d\tau \right\} \\ w_n &= \frac{1}{\lambda^2(\omega_{2n}^2 - \omega_{1n}^2)} \left\{ \frac{1}{\omega_{1n}} \int_0^t \text{Sin} \omega_{1n}(t-\tau) F_{nw}(\tau) d\tau \right. \\ &\quad \left. - \frac{1}{\omega_{2n}} \int_0^t \text{Sin} \omega_{2n}(t-\tau) F_{nw}(\tau) d\tau \right\} \end{aligned} \quad (4.51)$$

where

$$\begin{aligned} F_{n\psi}(t) &= B_2(2n+1) \left\{ \left( A_{3n} + \lambda \frac{d^2}{dt^2} \right) \int_0^1 P_n(\xi) \left[ \int_0^{\phi(\xi)} Q_{T1} d\phi \right. \right. \\ &\quad \left. \left. + H_{i\psi_n} \right] d\xi - [\alpha \Lambda_n + (1+\nu)] \int_0^1 [Q_{T2} + H_{iw_n}] P_n(\xi) d\xi \right\} \\ F_{nw}(t) &= B_2(2n+1) \left\{ \left[ \Lambda_n - (1-\nu) + \lambda \frac{d^2}{dt^2} \right] \int_0^1 [Q_{T2} + H_{iw_n}] P_n(\xi) d\xi \right. \\ &\quad \left. + \Lambda_n [\alpha \Lambda_n + (1+\nu)] \int_0^1 \left[ \int_0^{\phi(\xi)} Q_{T1} d\phi + H_{i\psi_n} \right] P_n(\xi) d\xi \right\} \end{aligned} \quad (4.52a)$$

and

$$\begin{aligned}
 H_{i\psi_m} &= \frac{1}{2n+1} \left\{ \left[ L + (1-\nu) - \lambda \frac{\partial^2}{\partial t^2} \right] G_{i\psi} \right. \\
 &\quad \left. + [\alpha L - (1+\nu)] G_{iw} \right\} \Big|_{n=m} \\
 H_{iw_m} &= \frac{1}{2n+1} \left\{ \left[ -LL - (1-\nu)\alpha L - 2(1+\nu) - \lambda \frac{\partial^2}{\partial t^2} \right] G_{iw} \right. \\
 &\quad \left. + [-\alpha LL + (1+\nu)L] G_{i\psi} \right\} \Big|_{n=m}
 \end{aligned} \tag{4.52b}$$

We shall now construct  $G_{1\psi}$ ,  $G_{1w}$ ,  $G_{2\psi}$ , and  $G_{2w}$  for each of the following boundary cases individually:

(A) Roller-hinged case

The boundary conditions to be satisfied are

$$\left. \begin{aligned}
 w &= 0 \\
 \frac{\partial u}{\partial \phi} &= B_2 a \theta_0^* \\
 \frac{\partial^2 w}{\partial \phi^2} &= -B_2 \left( \frac{\theta_1}{\alpha} + a \theta_0^* \right)
 \end{aligned} \right\} \text{ at } \phi = \frac{\pi}{2} \tag{4.53}$$

The second and third conditions result from the stress boundary conditions

$$\left. \begin{aligned} N_{\phi} &= 0 \\ M_{\phi} &= 0 \end{aligned} \right\} \text{ at } \phi = \frac{\pi}{2}$$

respectively.

We may satisfy the conditions in (4.53) by assuming

$$\psi = \sum_{n=1,3,5,\dots}^{\infty} \psi_n(t) P_n(x) + B_2 G_{1\psi} \quad (4.54)$$

$$= \sum_{n=1,3,5,\dots}^{\infty} \psi_n(t) P_n(x) - \frac{1}{2} B_2 a x \int_0^{\phi} \theta_0^* d\phi$$

$$w = \sum_{n=1,3,5,\dots}^{\infty} w_n(t) P_n(x) + B_2 G_{1w}$$

$$= \sum_{n=1,3,5,\dots}^{\infty} w_n(t) P_n(x) + \frac{1}{2} B_2 x \int_0^{\phi} \left( \frac{\theta_1^*}{\alpha} + a \theta_0^* \right) d\phi$$

(b) Roller-clamped case

The boundary conditions to be satisfied are

$$\left. \begin{aligned} u_{\phi} &= 0 \\ \frac{\partial w}{\partial \phi} &= 0 \\ Q &= 0 \end{aligned} \right\} \text{ at } \phi = \frac{\pi}{2} \quad (4.55)$$

In terms of displacements, the last of (4.55) reads

$$\frac{\partial^3}{\partial \phi^3} (\psi + w) = - \frac{B_2}{\alpha} \frac{\partial \theta_1^*}{\partial \phi} \quad \text{at } \phi = \frac{\pi}{2} \quad (4.56)$$

(i) For

$$\theta_1^* = \theta_1^*(t) \quad (4.57)$$

equation (4.56) becomes homogeneous. Now set

$$\psi = \sum_{n=2,4,6,\dots}^{\infty} \psi_n(t) P_n(x) \quad (4.58)$$

$$w = \sum_{n=0,2,4,\dots}^{\infty} w_n(t) P_n(x)$$

(ii) For

$$\theta_1^* = \theta_1^*(t, \phi) \quad (4.59)$$

(4.56) is satisfied if

$$\psi = \sum_{n=2,4,6,\dots}^{\infty} \psi_n(t) P_n(x) + B_2 G_{2\psi}(x, t) \quad (4.60)$$

$$= \sum_{n=2,4,6,\dots}^{\infty} \psi_n(t) P_n(x) - \frac{1}{12} \frac{B_2}{\alpha} \theta_1^* x^2$$

$$w = \sum_{n=0,2,4,\dots}^{\infty} w_n(t) P_n(x) + B_2 G_{2w}(x, t)$$

$$= \sum_{n=0,2,4,\dots}^{\infty} w_n(t) P_n(x) - \frac{1}{12} \frac{B_2}{\alpha} \theta_1^* x^2$$

### Numerical Examples

To provide a quantitative evaluation, numerical examples for the response of the apex of a hemispherical shell with a roller-clamped edge are presented in Figures 8 through 13 using the following types of loading:

(A) Exponentially decaying dynamic loading

$$p_\phi = 0 \quad (4.61)$$

$$p_n = e^{-bt} p_n(x)$$

with

$$a=3000 \text{ in.}, E = 30 \times 10^6 \text{ psi}, \nu = 0.33, \text{ and } \rho = 0.7298 \times 10^{-3} \text{ lb-sec}^2/\text{in}^4$$

(B) Constant temperature distribution

The temperature is the solution to the following boundary value problem [5,6]:

$$\frac{\partial^2 \theta^*}{\partial \zeta^2} = \frac{1}{K_d} \frac{\partial \theta^*}{\partial t} \quad t \geq 0 \quad |\zeta| \leq \frac{h}{2} \quad (4.62)$$

$$\frac{\partial \theta^*}{\partial \zeta} = \frac{1}{K_c} Q_0^* \quad t \geq 0 \quad \zeta = \frac{h}{2}$$

$$\frac{\partial \theta^*}{\partial \zeta} = 0 \quad t \geq 0 \quad \zeta = -\frac{h}{2}$$

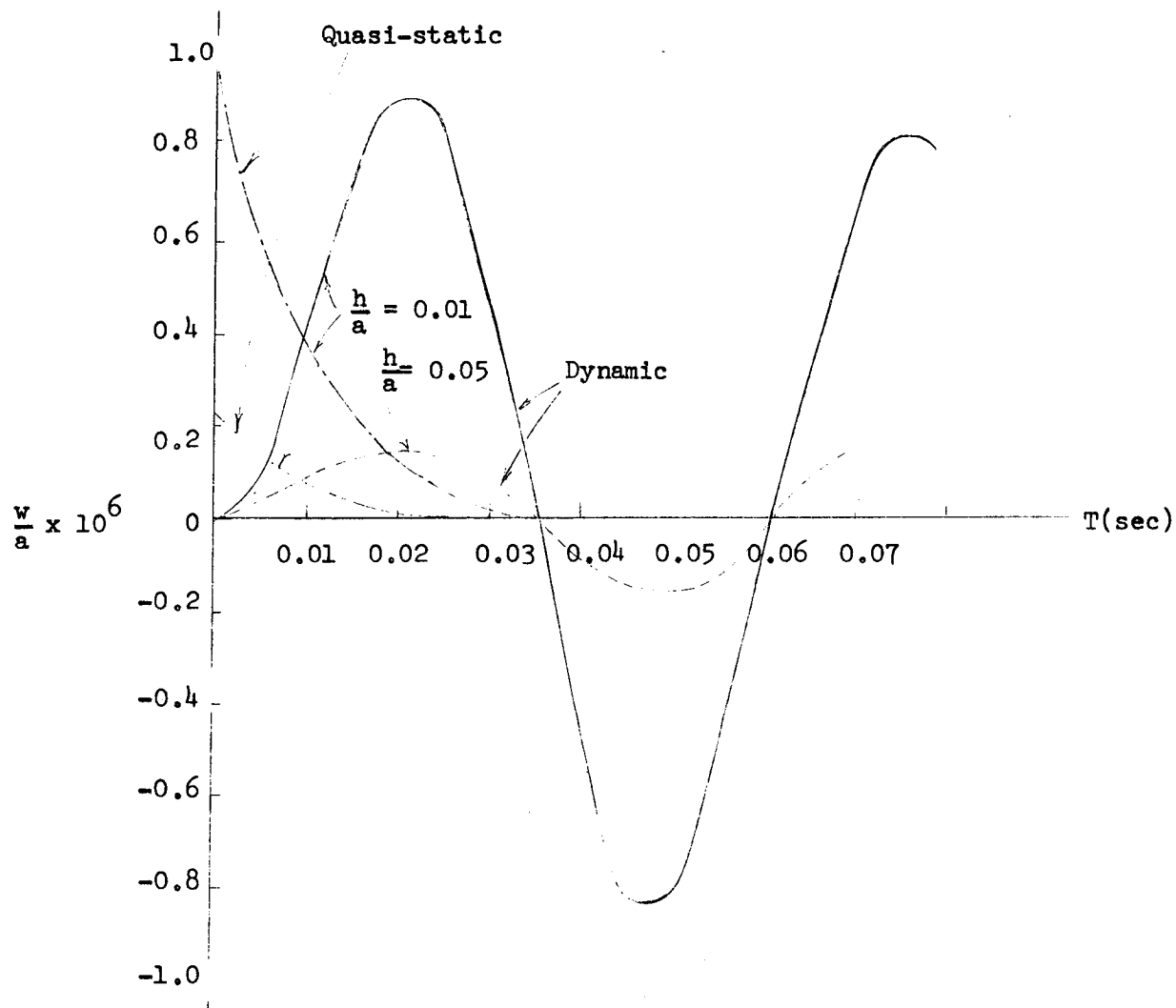


Figure 8. Displacement of Hemisphere at Apex Due to  
Uniformly Distributed Exponential Loading

$$p_n = e^{-100T}, \text{ Roller-Clamped Edge}$$

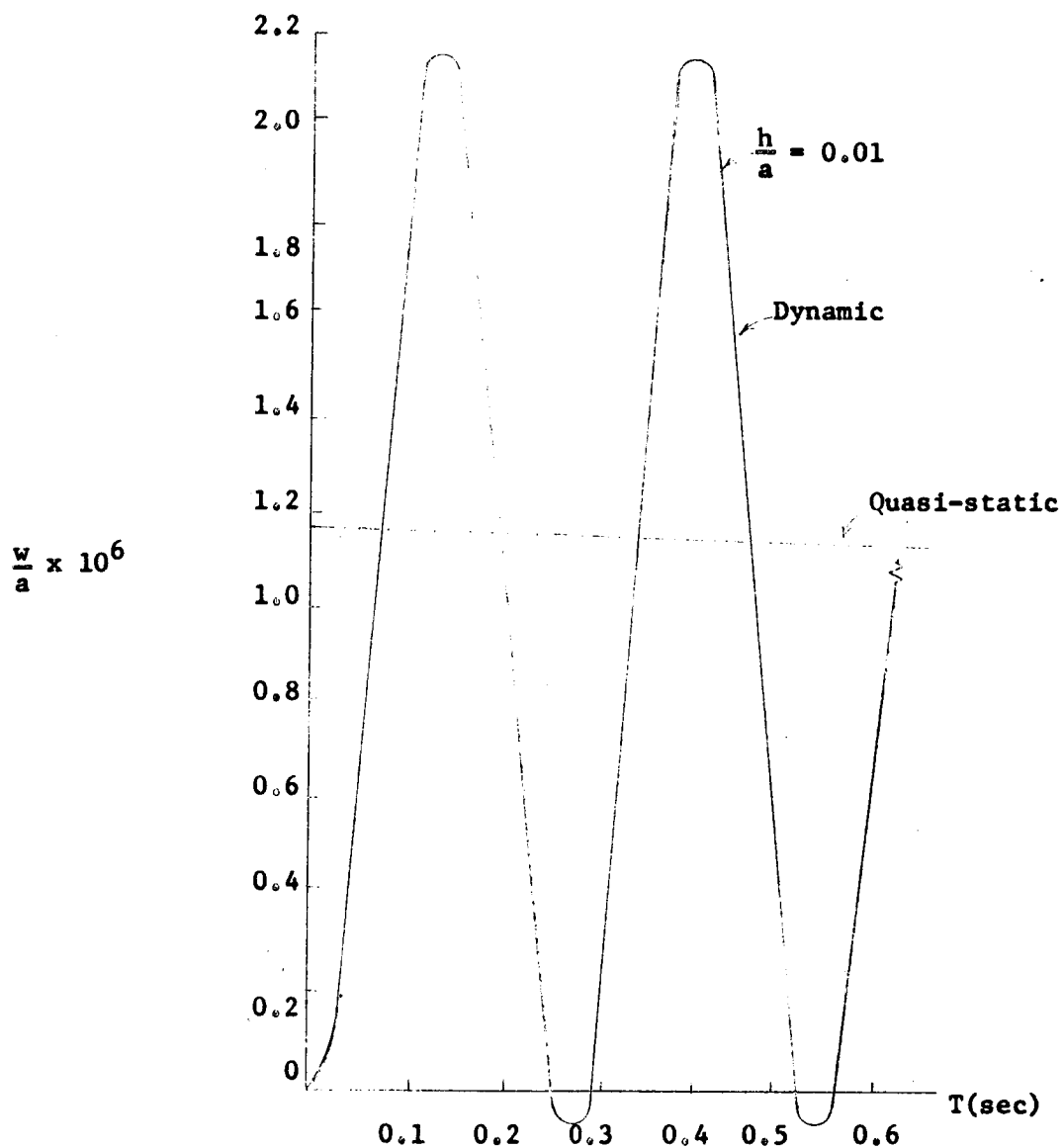


Figure 9. Displacement of Hemisphere at Apex  
 Due to Uniformly Distributed Exponential Loading  
 $p_n = e^{-0.25T}$ , Roller-Clamped Edge



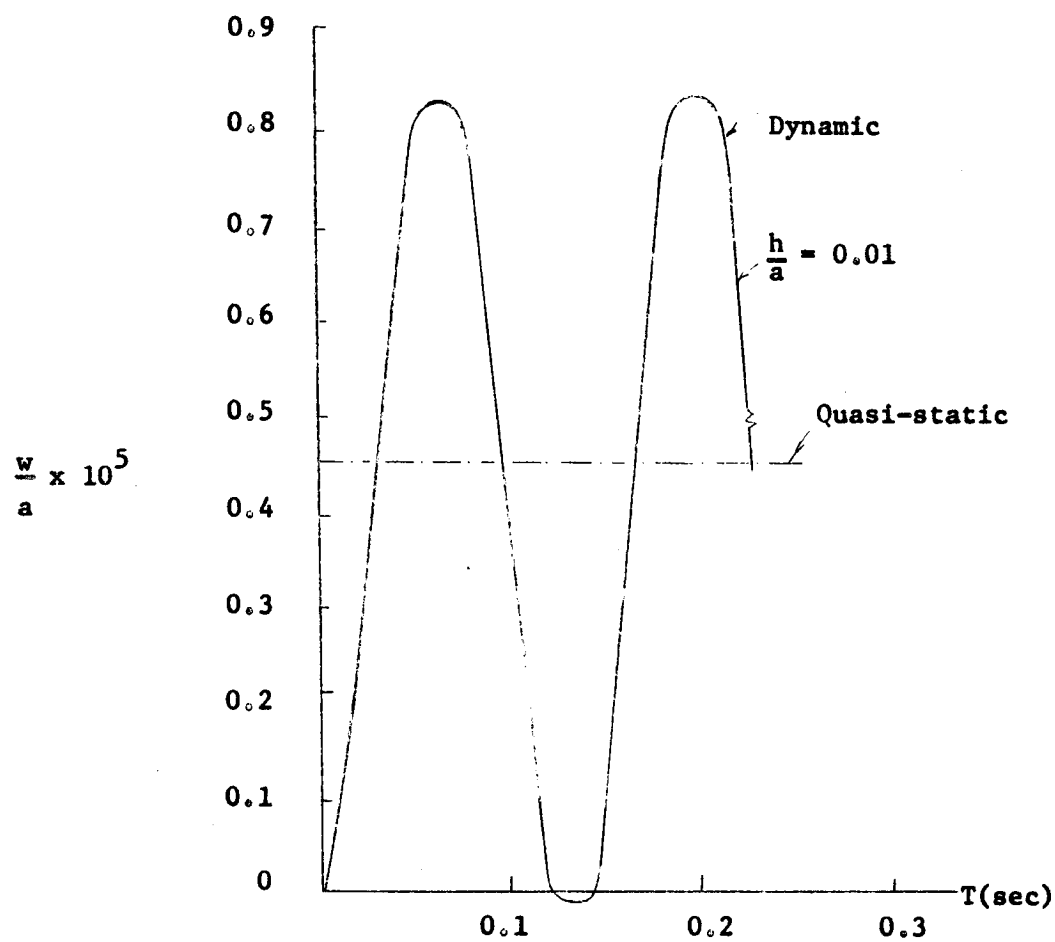


Figure 10. Displacement of Hemisphere at Apex  
Due to Uniformly Distributed Exponential Loading

$$p_n = P_2(x)e^{-0.25T}, \text{ Roller-Clamped Edge}$$

$$\theta^* = 0 \quad t = 0 \quad |z| \leq \frac{h}{2}$$

The solution for this problem is [7]

$$\begin{aligned} \theta^* = & \frac{Q_0^* K_d}{K_c h} t + \frac{h Q_0^*}{K_c} \left\{ \frac{1}{2} \left( \frac{z}{h} + \frac{1}{2} \right)^2 - \frac{1}{6} \right. \\ & \left. - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \exp \left[ -n^2 \pi^2 \frac{t K_d}{h^2} \right] \cos \left[ n \pi \left( \frac{z}{h} + \frac{1}{2} \right) \right] \right\} \end{aligned} \quad (4.63)$$

Finally, values of  $\theta_0^*$  and  $\theta_1^*$  may be calculated from equations (3.3). They are

$$\theta_0^* = R_4 t \quad (4.64)$$

$$\theta_1^* = R_1 - R_2 \sum_{n=1,3,\dots}^{\infty} \frac{1}{n^4} \exp \left[ -R_3 t n^2 \right]$$

with

$$\begin{aligned} R_1 = \frac{Q_0^* h^3}{24 K_c} \quad , \quad R_2 = R_1 \frac{96}{\pi^4} \\ R_3 = \pi^2 \frac{K_d}{h^2} \quad , \quad R_4 = \frac{Q_0^* K_d}{K_c} \end{aligned} \quad (4.65)$$

The results are plotted in Figures 11, 12, and 13. The symbol,  $B^*$ , shown in these figures is defined as

$$B^* = \left( \frac{D h^3}{K_d^2 a^4} \right)^{1/4}$$

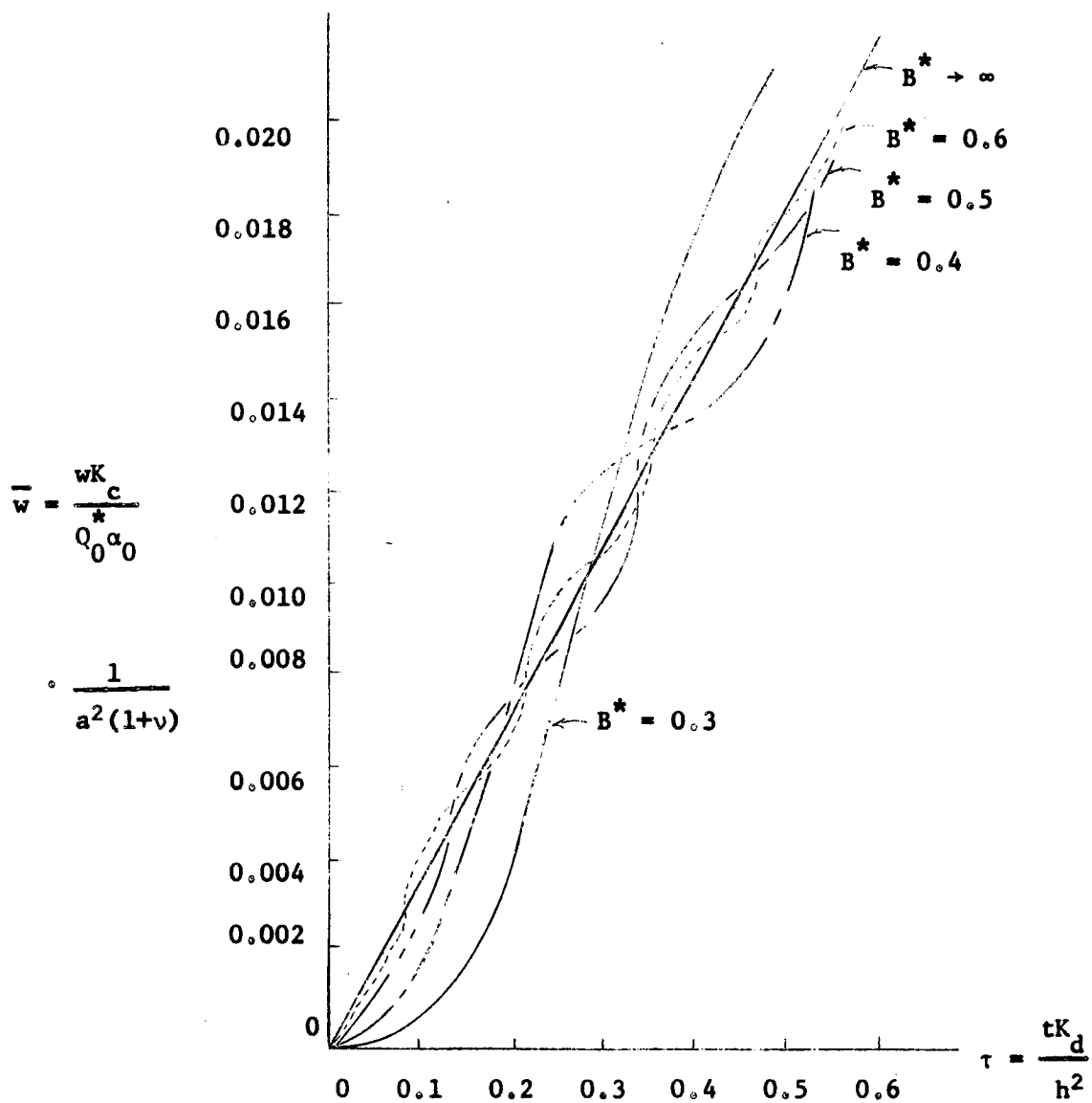


Figure 11. Displacement of Hemisphere at Apex Due to  
Uniformly Distributed Temperature,  
 $h/a = 0.05$ , Roller-Clamped Edge

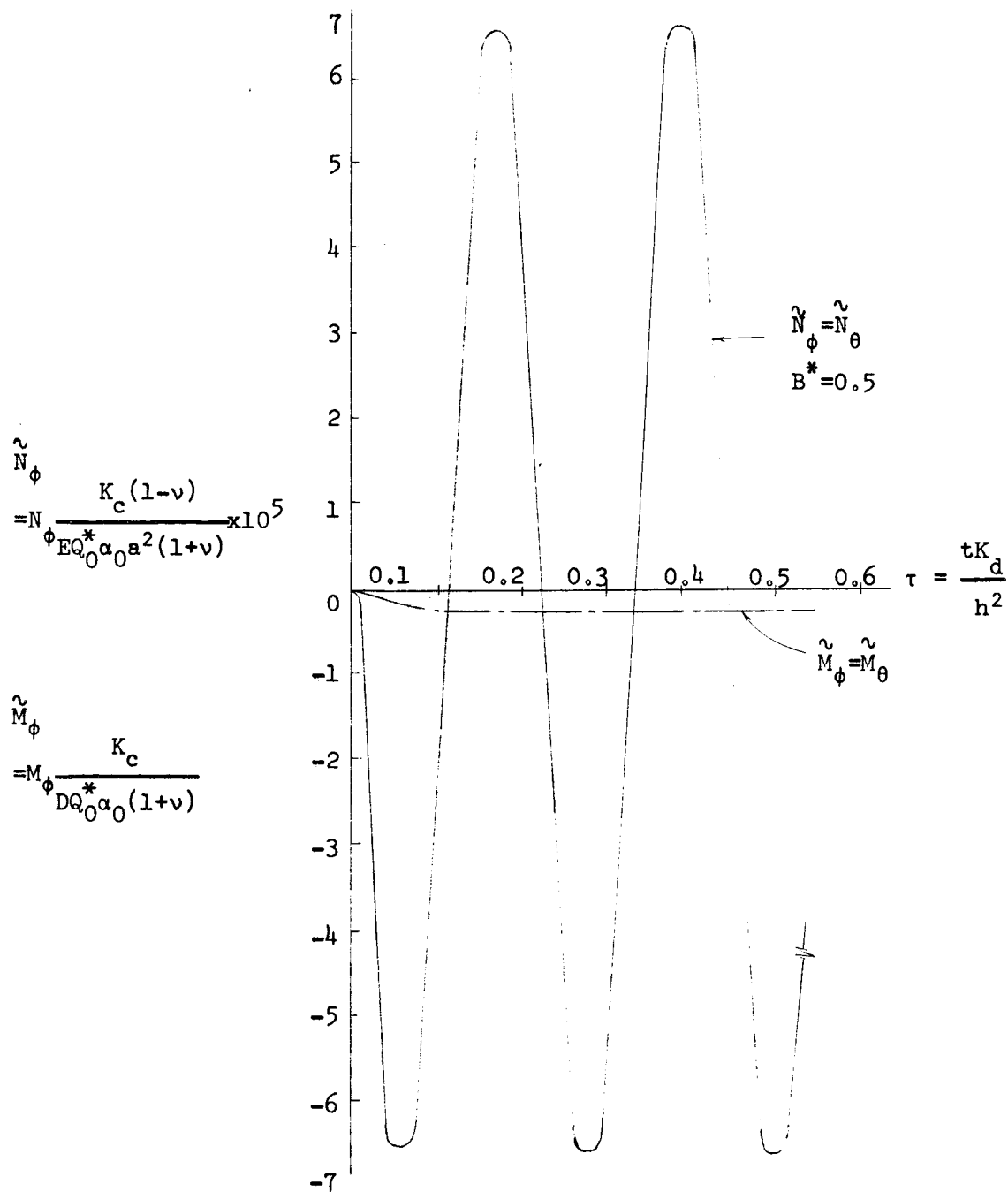


Figure 12. Stress and Moment of Hemisphere at Apex  
 Due to Uniformly Distributed Temperature,  
 $h/a = 0.05$ , Roller-Clamped Edge

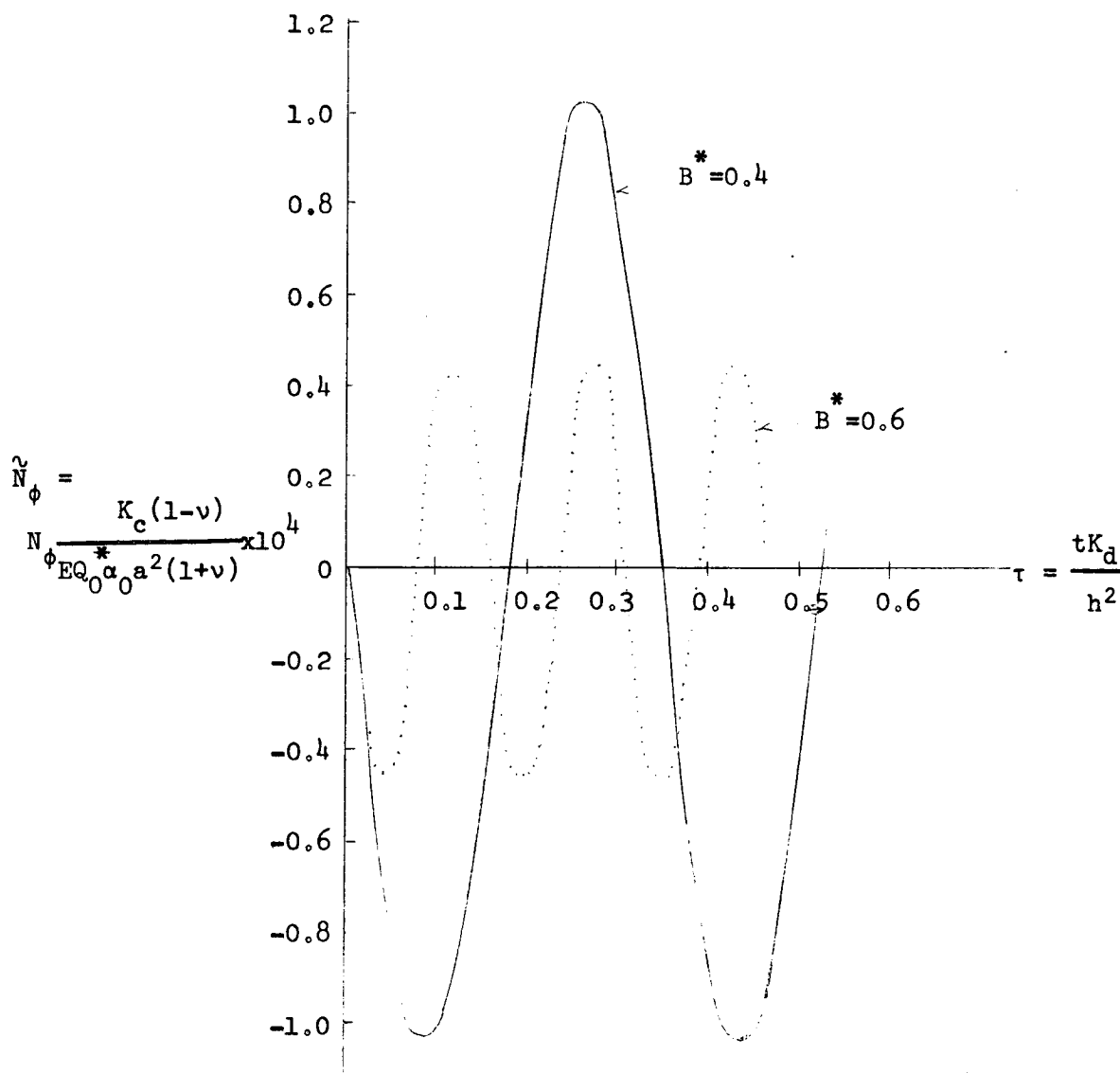


Figure 13. Thermal Stress of Hemisphere at Apex

Due to Uniformly Distributed Temperature,

$h/a = 0.05$ , Roller-Clamped Edge

### Conclusions

(A) The variation of shell thickness has very little effect on the upper branch frequencies for hemispherical shells having roller-clamped edges and roller-hinged edges. However, lower branch frequencies are affected significantly as the shell thickness increases. Furthermore, the lower branch frequencies for roller-clamped and roller-hinged shells are not bounded. This contradicts the results obtained according to membrane theory given in [8].

(B) The natural frequencies of a shell completely clamped approach the frequencies of a roller-clamped shell when the thickness of the shell decreases.

(C) The response of a hemispherical shell to an exponentially decaying dynamic loading shows that, except for loading with rather short time duration, quasi-static response in general provides an average response. Hence when an exact solution can be obtained for the quasi-static part, the mode-acceleration method has a definite advantage over the usual modal analysis.

(D) The response of a shell subjected to a uniformly distributed temperature input shows that the quasi-static results never reach a definite value but increase with respect to time; and the dynamic solution oscillates about the quasi-static value.

(E) In the present situation, the thermal conditions lead, according to equation (4.64), to an in-plane force which is linear in time, and to a thermal moment that reaches its steady state in a

short time period. The thermal response solution thus has the tendency to oscillate closely about the corresponding quasi-static solution.

(F) The thermal stress obtained in the roller-clamped shell is the evidence of the existence of thermally induced vibration. The customary quasi-static thermal problem for this case displays no thermal stress response. It gives only a free expansion of the shell.

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## CHAPTER V

## AXISYMMETRIC VIBRATION OF AN ELASTIC CYLINDER

## WITH A HEMISPHERICAL SHELL BOTTOM

Recent investigations of shell vibrations are limited mostly to shells of specific, simple configurations such as cylindrical and spherical. The dynamic analysis of cylindrical shells can be found in [1,2,3]. The dynamic analysis of spherical shells can be found in [4,5,6]. For the analysis of composite shells we mention here that Coale and Nagano [7] have considered the flexibility of a cylindrical-hemispherical tank for the analysis of liquid sloshing. Only a membrane theory was used.

The present study is concerned with the axisymmetric vibration of a cylindrical shell with a hemispherical shell bottom. The equations resulting from linear bending theory are used.

General Formulation and Approach

Due to the difference in the geometric structure of the shell components, it is advantageous to use different coordinate systems for each portion of the shell. The geometry and the coordinate systems are shown in Figure 14. The equations of motion for each part may be written in the following general forms:

$$[L_{\alpha\beta}] \{u_i\} + \rho h \{\ddot{u}_i\} = \{q_{u_i}\} \quad (5.1)$$

and

$$L_{\alpha\beta}^* \{u_i^*\} + \rho^* h^* \{\ddot{u}_i^*\} = 0 \quad (5.2)$$

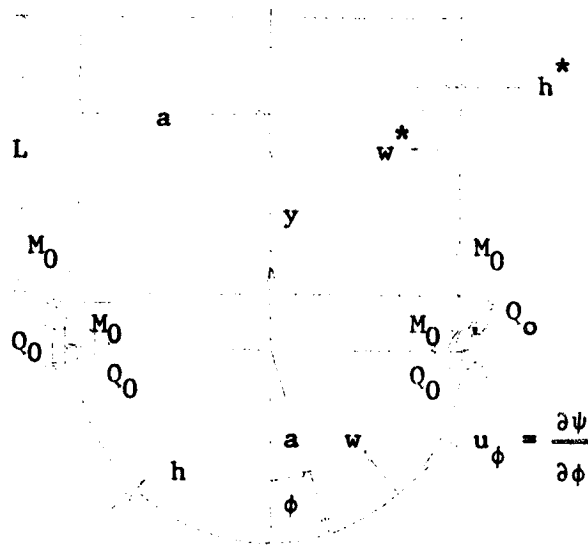


Figure 14. Geometry and the Edge Effects

where  $L_{\alpha\beta}$  and  $L_{\alpha\beta}^*$  are spatial, differential operators,  $u_i$  and  $u_i^*$  are displacements and the  $q_{u_i}$  are the load components, which include the unknown moment  $M_0$  and unknown shear  $Q_0$  arising at the junction of the two shell configurations.

The motion of the system is assumed to be harmonic, or

$$u_i = \bar{U}_i e^{i\omega t} \quad (5.3)$$

$$M_0 = \bar{M} e^{i\omega t}$$

$$Q_0 = \bar{Q} e^{i\omega t}$$

First, the general solution for each portion is obtained, and all boundary conditions except those at the junction are satisfied. The support conditions at the junction are shown in Figure 14. The requirement that the deformation be compatible along the junction of the shell segments then results in the homogeneous set of algebraic equations

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \bar{Q} \\ \bar{M} \end{bmatrix} = \{0\} \quad (5.4)$$

involving the unknown amplitudes  $\bar{Q}$  and  $\bar{M}$  of the shear and moment, respectively. The elements  $a_{ij}$  of the coefficient matrix contain the frequency  $\omega$  as a parameter. The frequency equation is obtained by equating the coefficient determinant to zero. The numerical determination of the frequencies is accomplished by means of an iterative procedure.

#### The Cylindrical Shell

The equation for the axisymmetric motion of a cylindrical shell, when the longitudinal inertia is neglected and when the in-plane axial force is zero, is [1]

$$\frac{\partial^4 w^*}{\partial y^4} + \frac{E^* h^*}{D^* a^2} w^* + \frac{\rho^* h^*}{D^*} \frac{\partial^2 w^*}{\partial t^2} = 0 \quad (5.5)$$

The geometry and terminology are apparent from Figure 14. The assumption of harmonic motion of the form

$$w^*(y, t) = \bar{w}(y)e^{i\omega t} \quad (5.6)$$

results in

$$\frac{d^4 \bar{w}}{dy^4} - \lambda^{*4} \bar{w} = 0 \quad (5.7)$$

The solution of equation (5.7) is

$$w(y) = A \cosh \lambda^* y + B \sinh \lambda^* y + C \cos \lambda^* y + D \sin \lambda^* y \quad (5.8)$$

with

$$\lambda^{*4} = \frac{\rho^* h^*}{D^*} \omega^2 - \frac{E^* h^*}{D^* a^2}$$

Two of the integration constants may be expressed in terms of the other two constants by using the boundary conditions along  $y = l$ , with the result

$$C = \alpha(l)A + \beta(l)B \quad (5.9)$$

$$D = \gamma(l)A + \zeta(l)B$$

where, for a shell clamped at  $y = l$

$$\alpha(l) = -\cosh \lambda^* l \cos \lambda^* l + \sinh \lambda^* l \sin \lambda^* l$$

$$\beta(l) = -\sinh \lambda^* l \cos \lambda^* l + \cosh \lambda^* l \sin \lambda^* l$$

$$\begin{aligned}\gamma(l) &= -\sinh \lambda^* l \cos \lambda^* l - \cosh \lambda^* l \sin \lambda^* l \\ \zeta(l) &= -\cosh \lambda^* l \cos \lambda^* l - \sinh \lambda^* l \sin \lambda^* l\end{aligned}$$

The boundary conditions along  $y = 0$  are

$$M_{yy}(0, t) = M_0(0)e^{i\omega t} \quad (5.10)$$

$$Q(0, t) = Q_0(0)e^{i\omega t}$$

The application of equations (5.10) to (5.8) in conjunction with equations (5.9) results in the general solution for the vibration of the cylindrical part in terms of the unknowns  $M_0$  and  $Q_0$ , i.e.

$$\begin{aligned}w(y, t) &= M_0(0)e^{i\omega t} [C_{11}f(\lambda^* y) + C_{22}g(\lambda^* y)] \\ &+ Q_0(0)e^{i\omega t} [C_{12}f(\lambda^* y) + C_{21}g(\lambda^* y)]\end{aligned} \quad (5.11)$$

where

$$f(\lambda^* y) = \cosh \lambda^* y + \alpha(l) \cos \lambda^* y + \gamma(l) \sin \lambda^* y \quad (5.12)$$

$$g(\lambda^* y) = \sinh \lambda^* y + \beta(l) \cos \lambda^* y + \zeta(l) \sin \lambda^* y$$

and

$$C_{11} = -\frac{1-\zeta(\ell)}{\Delta\lambda^{*2}}, \quad C_{12} = -\frac{\beta(\ell)}{\Delta\lambda^{*3}} \quad (5.13)$$

$$C_{21} = -\frac{1-\alpha(\ell)}{\Delta\lambda^{*3}}, \quad C_{22} = -\frac{\gamma(\ell)}{\Delta\lambda^{*2}}$$

$$\Delta = D^* [1 - \alpha(\ell) - \zeta(\ell) + \alpha(\ell)\zeta(\ell) - \gamma(\ell)\beta(\ell)]$$

### Hemispherical Bulkhead

The equations of motion are those of (4.1)

$$L(\psi) = -\alpha L(w) + (1+\nu)w - (1-\nu)\psi \quad (5.14)$$

$$LL(w) = -LL(\psi) + (1-\nu)L(w) + \frac{1+\nu}{\alpha} L(\psi) \\ - \frac{2(1+\nu)}{\alpha} w - \frac{\lambda}{\alpha} \frac{\partial^2 w}{\partial t^2} + \frac{\lambda}{\alpha \rho h} p_n$$

where the in-plane inertia term is neglected and

$$\alpha = \frac{1}{12} \frac{h}{a}^2, \quad \lambda = \rho \frac{a^2(1-\nu^2)}{E} \quad (5.15)$$

$$L(\cdot) = (1-x^2) \frac{\partial^2(\cdot)}{\partial x^2} - 2x \frac{\partial(\cdot)}{\partial x} \quad \text{and} \quad x = \cos \phi$$

For  $w$  the boundary conditions corresponding to the simply supported case are prescribed. Considering the motion of the shell to be harmonic,

$$\psi(x, t) = e^{i\omega t} \sum_{n=0}^{\infty} B_n P_n(x) \quad (5.16)$$

$$w(x, t) = e^{i\omega t} \sum_{n=0}^{\infty} A_n P_n(x)$$

Equations (5.14) then become

$$- [(1-\nu)] - \Lambda_n] B_n + [\alpha\Lambda_n + (1+\nu)] A_n = 0 \quad (5.17)$$

$$\Lambda_n [(1+\nu) + \alpha\Lambda_n] B_n + \{-\lambda\omega^2 + 2(1+\nu)$$

$$+ \alpha\Lambda_n [\Lambda_n - (1-\nu)]\} A_n$$

$$= - (2n+1) \frac{\lambda}{\rho h} \int_0^1 p_n P_n(\xi) d\xi$$

where  $\Lambda_n = n(n+1)$ . The transverse loading may be written as

$$p_n = Q_0 \{\xi-0\}^{-1} - M_0 \{\xi-0\}^{-2} \quad (5.18)$$

Here  $\{x-0\}^{-1}$  and  $\{x-0\}^{-2}$  are singular functions resulting in

$$\int_0^1 p_n P_n(\xi) d\xi = Q_0 P_n(0) + M_0 P_n'(0) \quad (5.19)$$

where the prime denotes differentiation with respect to  $\xi$ . The substitution of equation (5.19) into equation (5.18) and the subsequent simultaneous solution of equations (5.17) result in the expressions



$$A_n = -a_{1n} [P_n(0) Q_0 + P_n'(0) M_0] \quad (5.20)$$

$$B_n = a_{2n} A_n$$

for the coefficients  $A_n$  and  $B_n$ . Here

$$a_{1n} = \frac{1}{K_h} [(1-\nu) - \Lambda_n] (2n+1) \quad (5.21)$$

and

$$a_{2n} = \frac{2\Lambda_n + 1+\nu}{1-\nu - \Lambda_n}$$

with

$$K_h = \rho h \{-\lambda \omega^2 + 2(1+\nu) - \alpha \Lambda_n [\Lambda_n - (1-\nu)]^2 + \Lambda_n [(1+\nu) + \alpha \Lambda_n]^2\} \quad (5.22)$$

Finally, the substitution of equations (5.20) into equations (5.16) gives

$$w(x, t) = -e^{i\omega t} \sum_{n=0}^{\infty} a_{1n} P_n(x) [Q_0 P_n(0) + M_0 P_n'(0)] \quad (5.23)$$

$$\psi(x, t) = -e^{i\omega t} \sum_{n=0}^{\infty} a_{1n} a_{2n} P_n(x) [Q_0 P_n(0) + M_0 P_n'(0)]$$

### Frequency Equations

As indicated in the general discussion, the frequency equation for the system as a whole may be obtained by requiring compatible

deformation along the junction of the two shell segments. The conditions to be satisfied are:

$$w^*(0, t) = + w(0, t) \quad (5.24)$$

and

$$\frac{\partial w^*(0, t)}{\partial x} = \frac{1}{a} \frac{\partial w(\frac{\pi}{2}, t)}{\partial \phi}$$

The imposition of these conditions on equations (5.11) and (5.23) results in

$$M_0 [C_{11}f(0) + C_{22}g(0)] + Q_0 [C_{12}f(0) + C_{21}g(0)] \quad (5.25)$$

$$+ \sum_{n=0}^{\infty} a_{1n} P_n^2(0) \Big] = 0$$

$$M_0 [C_{11}f'(0) + C_{22}g'(0)] + Q_0 [C_{12}f'(0) + C_{21}g'(0)]$$

$$- \frac{1}{a} \sum_{n=0}^{\infty} a_{1n} P_n'^2(0) \Big] = 0$$

Nontrivial solutions for  $M_0$  and  $Q_0$  exist if, and only if, the coefficient determinant for equations (5.25) vanishes. The resulting frequency equation is

$$[C_{11}f(0) + C_{22}g(0)] [C_{12}f'(0) + C_{21}g'(0)] \quad (5.26)$$

$$- \left\{ c_{11} f'(0) + c_{22} g'(0) - \frac{1}{a} \sum_{n=0}^{\infty} a_{1n} [P_n'(0)]^2 \right\} \cdot$$

$$\left[ c_{12} f(0) + c_{21} g(0) + \sum_{n=0}^{\infty} a_{1n} P_n^2(0) \right] = 0$$

### Numerical Example

The natural frequencies are calculated for various thicknesses and lengths of the cylindrical part of the configuration. The following dimensions and material properties were used for both the bulkhead and the cylindrical part of the system:

$$E = 30 \times 10^6 \text{ psi}, a = 20 \text{ in.}, \nu = 1/3, \rho = 0.00735 \text{ lb-sec}^2/\text{in}^4$$

The open end of the shell is considered as clamped. The resulting frequencies are compared to those of a clamped-clamped cylindrical shell [1]. The results are shown in Tables 2 and 3. The natural frequencies corresponding to a clamped-clamped cylinder are denoted by  $\omega_n$ .  $\Omega_n$  represents the frequencies corresponding to the present numerical example.

The effect due to the bending rigidity of the cylinder and the flexibility of the bottom on the fundamental frequency of the cylinder may be shown by comparing  $\Omega_1$  to  $\sqrt{\frac{E}{\rho a^2}}$  and  $\omega_n$  to  $\Omega_n$ , where  $\sqrt{\frac{E}{\rho a^2}}$  is the membrane frequency of a cylinder. The comparison of  $\omega_n$  and  $\Omega_n$  is illustrated in Figures 15 and 16. A typical curve of

$$S = |a_{ij}|$$

Table 2. Natural Frequencies for L = 40 Inches

Frequencies	Modes	1st Mode	2nd Mode	3rd Mode
h = 1.0 in.	$\Omega^2$	$0.10239 \times 10^9$	$0.10569 \times 10^9$	$0.11750 \times 10^9$
	$\omega^2$	$0.10279 \times 10^9$	$0.10775 \times 10^9$	$0.12402 \times 10^9$
h = 0.4 in.	$\Omega^2$	$0.10210 \times 10^9$	$0.10260 \times 10^9$	$0.10441 \times 10^9$
	$\omega^2$	$0.10216 \times 10^9$	$0.10296 \times 10^9$	$0.10556 \times 10^9$
h = 0.2 in.	$\Omega^2$	$0.10206 \times 10^9$	$0.10219 \times 10^9$	$0.10268 \times 10^9$
	$\omega^2$	$0.10207 \times 10^9$	$0.10227 \times 10^9$	$0.10292 \times 10^9$

Table 3. Natural Frequencies for H = 0.4 Inches

Frequencies	Modes	1st Mode	2nd Mode	3rd Mode
L = 20 in.	$\Omega^2$	$0.10287 \times 10^9$	$0.10811 \times 10^9$	$0.12354 \times 10^9$
	$\omega^2$	$0.10395 \times 10^9$	$0.11381 \times 10^9$	$0.15791 \times 10^9$
L = 40 in.	$\Omega^2$	$0.10210 \times 10^9$	$0.10260 \times 10^9$	$0.10441 \times 10^9$
	$\omega^2$	$0.10216 \times 10^9$	$0.10300 \times 10^9$	$0.10556 \times 10^9$
L = 60 in.	$\Omega^2$	$0.10205 \times 10^9$	$0.10217 \times 10^9$	$0.10254 \times 10^9$
	$\omega^2$	$0.10206 \times 10^9$	$0.10222 \times 10^9$	$0.10274 \times 10^9$

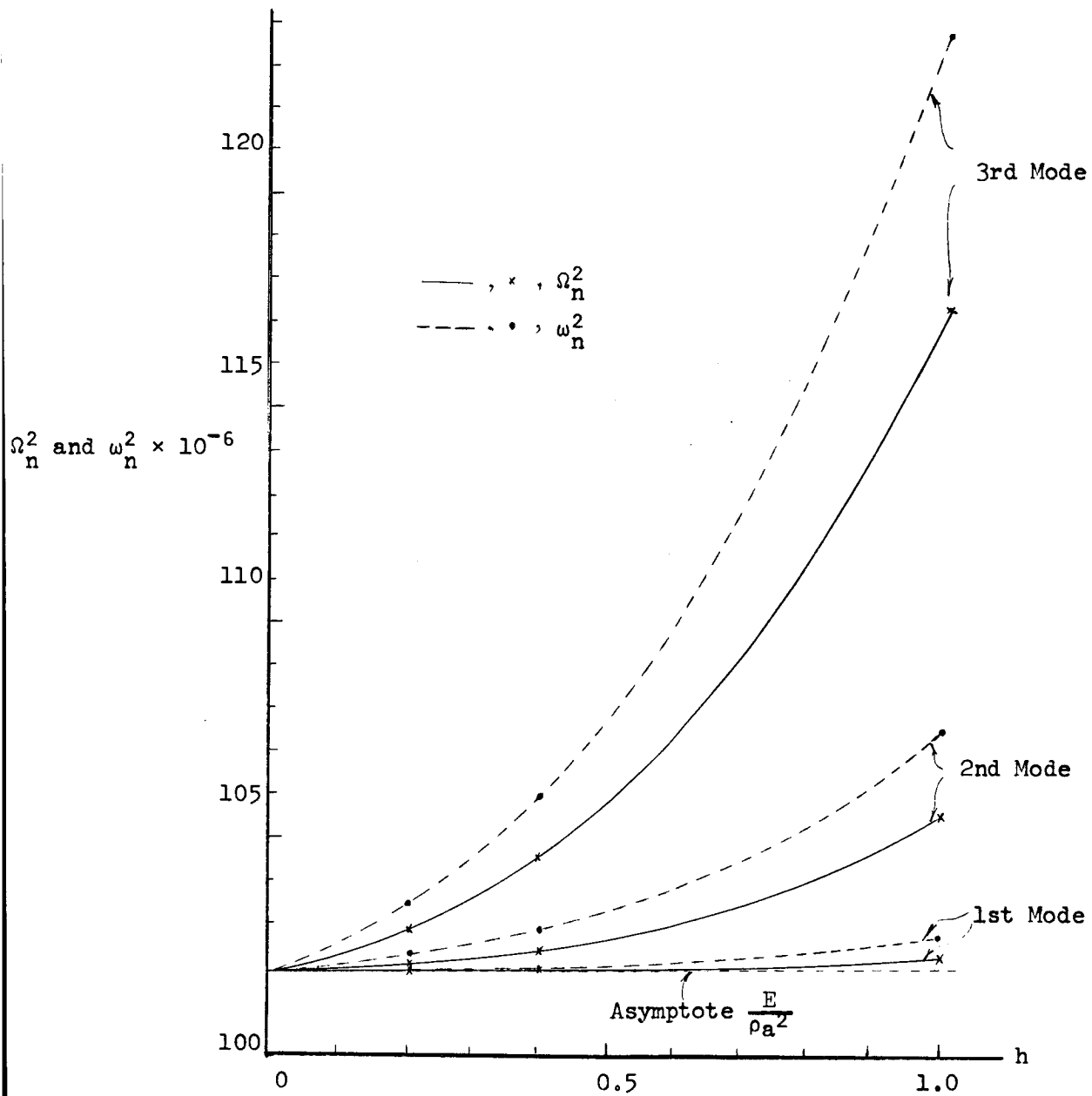


Figure 15. Comparison of  $\Omega_n^2$  and  $\omega_n^2$ ,

$L = 40$  in.

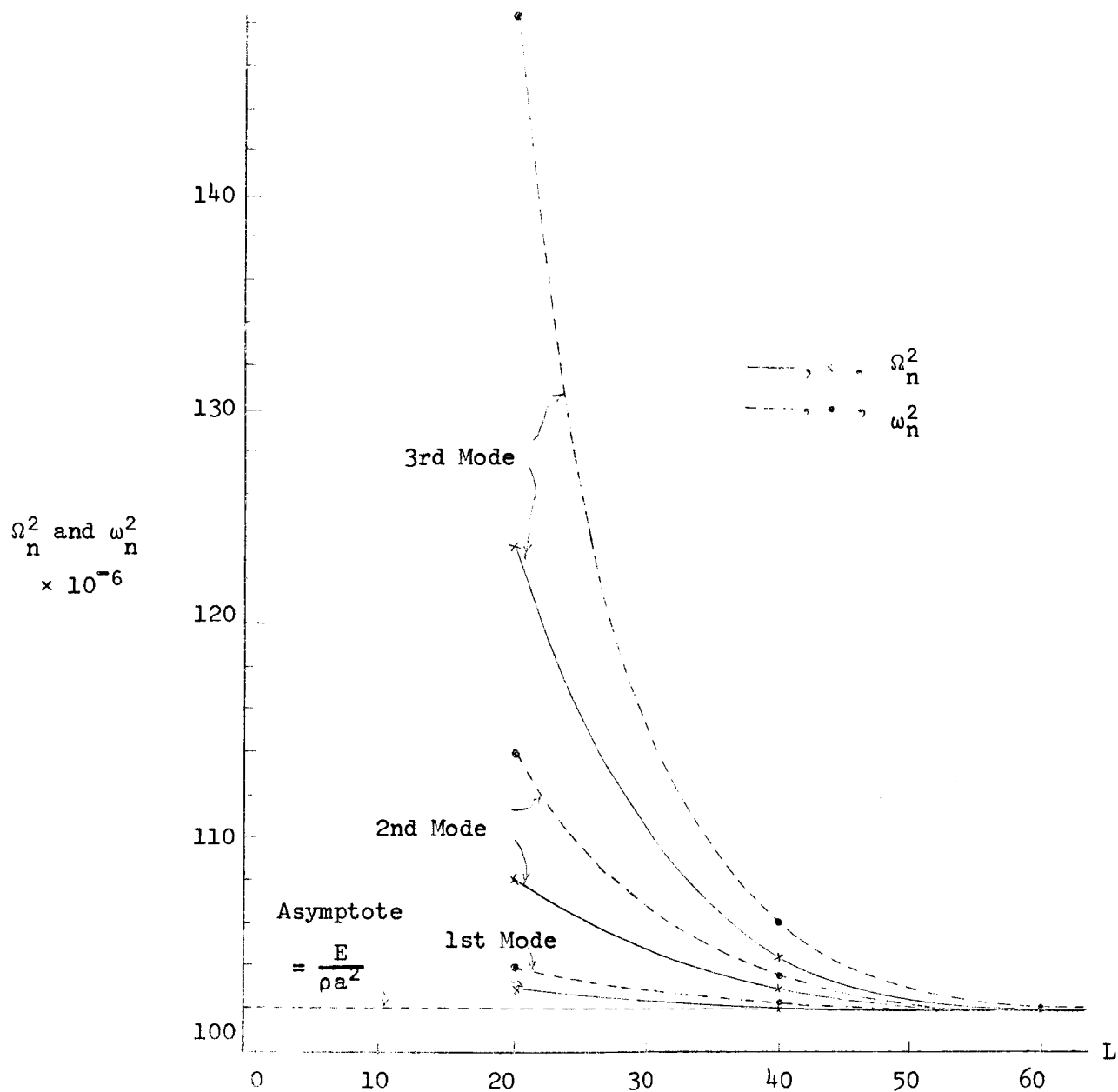


Figure 16. Comparison of  $\Omega_n^2$  and  $\omega_n^2$ ,  
 $h = 0.4$  in.

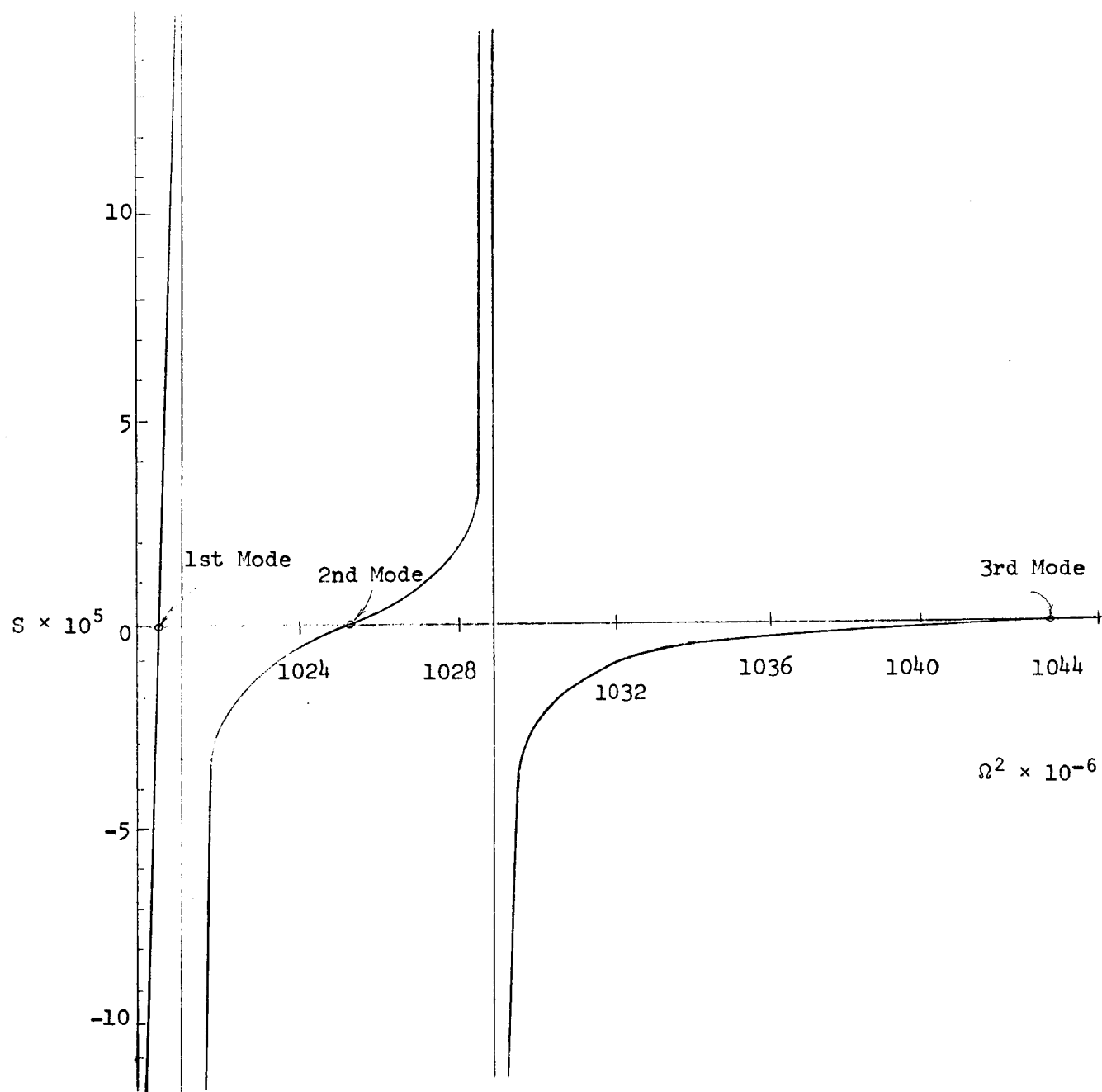
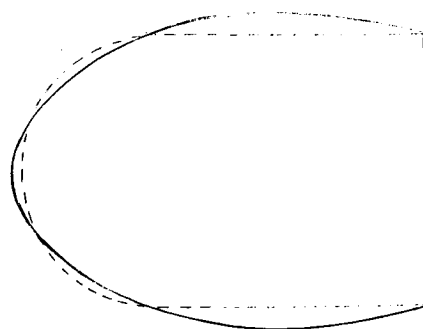


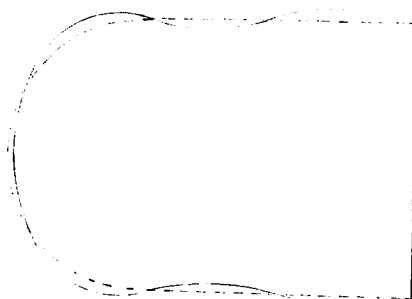
Figure 17. Typical Curve for  $S$  vs  $\Omega^2$  ,

$h = 0.4$  in. and  $L = 40$  in.





1st Mode



2nd Mode



3rd Mode

Figure 18. Mode Shapes

versus  $\Omega^2$  is shown in Figure 17. Figure 18 shows the first three mode-shapes.

### Conclusions

1. The frequencies for an elastic cylinder with a hemispherical bottom are obtained in accordance with the bending theory of elastic thin shells. Results indicate the changes in magnitude of the shell system frequencies closely follow the pattern of a clamped-clamped cylinder.

2. The numerical results also show that the difference in the clamped-clamped and the present composite shell frequency is greatest for short cylinders and decreases as the length of the cylinder increases as shown in Figure 15.

3. For very thin shells, the frequencies of the shell system are close to those of a cylinder without bottom, and the fundamental frequencies reduce to the quantity  $\sqrt{\frac{E}{\rho a^2}}$  as the thickness tends to zero for both cases under consideration. (See Figure 16).

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## CHAPTER VI

## RANDOM EXCITATION OF THIN ELASTIC SHELLS

In 1905, Einstein [1] studied the Brownian motion of a free particle and obtained the mean-square value of the displacement of the particle. Uhlenbeck and Ornstein [2] developed the theory of Brownian motion in 1930. Here the mean values of all the powers of the velocity and the displacement of a free particle in Brownian motion have been calculated, and the velocity of a harmonic oscillator has been obtained by using the Fokker-Planck equation. While most of the work in the earlier stage is concerned with the Brownian motion of a one degree-of-freedom system, Van Lear and Uhlenbeck [3] applied the method introduced by Ornstein and Uhlenbeck to calculate the Brownian motion mean-square deviation for strings and for elastic rods. In 1945, Wang and Uhlenbeck [4] developed the theory of reference [2] by using the theory of Gaussian random process. Also in this same paper the contributions to the theory of random vibrations previously accomplished have been summarized.

The first investigation on the buffeting problem by using statistical concepts was done by Liepmann [5] in 1952. The response of strings to random noise fields was studied and compared with some experimental results by Lyon [6]. Eringen [7], first obtained the response of beams and plates to random loads in 1956. His work includes the cases of simply supported bars, cantilever bars, clamped

plates, and simply supported rectangular plates. Samuels and Eringen [8] in 1958 studied the response of a simply supported, damped Timoshenko beam to a purely random Gaussian process. Caughey [9] has obtained the result that the mean-square deflection at every point of a nonlinear string is smaller than that for the equivalent linear string. Crandall and Yildiz [10] have studied the random vibration of beams by using several different dynamic models such as the Bernoulli-Euler beam, the Timoshenko beam, the Rayleigh beam, and a beam which has the shear flexibility of the Timoshenko beam but not the rotatory inertia. Y. K. Lin [11] has investigated the response of a nonlinear flat panel under periodic and random excitation on the assumption of a dominant fundamental mode. Caughey and Stumpf [12] analyzed the transient response of a simple harmonic oscillator to a stationary random input having an arbitrary power spectrum and applied its solution to the application of earthquake problems.

The response to white noise excitation of a light elastic string loaded at equal intervals by a number of equal masses is examined by Ariaratnam [13] using the theory of the Markov random process and the associated Fokker-Planck equation. Caughey [14] derived in 1963 the Fokker-Planck equation starting with the basic concepts of probability theory and then applied this to discrete nonlinear dynamic systems subjected to white random excitation. By using the theory of Markov processes and the associated Fokker-Planck equation, the random vibrations of a hinged, axially restrained, nonlinear elastic beam has been studied by Herbert [15], and the same approach has been used by the same author in studying the random

vibrations of plates with large amplitudes [16].

The progress achieved by the large number of investigators who have studied the launch-vehicle buffeting problem in the four years of its recognized existence since 1961 up to 1965 has been reviewed by Rainey [17]. In [18], the response of a simple oscillator to separable nonstationary random noise has been studied by MacNeal, Barnoski, and Bailie. The effects of transonic buffeting on a hammer head-shaped payload has been studied by Andrews [19] in 1966. Recently, Peterson, Howard, and Philippus [20] have studied the response of launch vehicles to separable nonstationary random transonic buffeting excitation.

The study of the vibration of thin shells acted on by broadband stationary random loads has been done by several authors [21, 22, 23]. The asymptotic method is used. The purpose of this investigation is to study the response of a thin elastic shell to separable nonstationary random loadings. Using a matrix differential operator, the general equations of motion for a torsionless arbitrary shell are written in vector form. The effect of a viscoelastic foundation is included. The complete solution to the transient vibration problem is then sought by using the method of spectral representation for the unknown variables. Once this solution is obtained, the dynamic response of the whole system may be obtained in a convolution integral. The statistical values for the shell, when the shell is considered to be excited by some random processes, may be calculated. For the purpose of illustration, hemispherical shells with roller-

hinged and roller-clamped edges are analyzed when the pressures are either broadband processes or band-limited processes.

### Dynamic Response of an Arbitrary Shell

The general equations of motion of a shell, when the effect of a viscoelastic foundation is included, may be written as a system of three linear, coupled differential equations of the form\*

$$[L_{\alpha\beta}] \{u_\beta\} = \{F_\alpha\}^{**} \quad (6.1)$$

where  $[L_{\alpha\beta}]$  is a matrix differential operator and where  $\{u_\beta\}$  and  $\{F_\alpha\}$  are the displacement and generalized load column vectors, respectively. Further

$$\{F_\alpha\} = k_f u_\alpha + \lambda_f \frac{\partial u_\alpha}{\partial t} + \rho h \frac{\partial^2 u_\alpha}{\partial t^2} - p_\alpha \quad (6.2)$$

The effect of a viscoelastic foundation, characterized by an elastic parameter  $k_f$  and a viscous damping parameter  $\lambda_f$ , has been included in equation (6.2) with the assumption that  $k_f$  as well as  $\lambda_f$  is the same in the normal and tangential directions of the coordinate curves.  $\{p_\alpha\}$  is the load column vector.

We employ the method of spectral representation for the unknown variables. Designating by  $u_\alpha$  any dependent variable of a solution state of equation (6.1), we express the variables in the form

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\*Detailed formulation of this section may be seen in Reference [24].

\*\*All Greek indices range from 1 to 3, and the Roman indices from 1 to  $\infty$  unless specified otherwise.

$$u_{\alpha}(\xi_1, \xi_2, t) = \sum_{i=1}^{\infty} q_i(t) U_{i\alpha}(\xi_1, \xi_2) \quad (6.3)$$

where  $q_i(t)$  are the generalized coordinates and  $U_{i\alpha}$  designates the dependent variables of a mode of undamped free vibration with a natural frequency  $\omega_i$ , and  $\xi_1$  and  $\xi_2$  are the coordinates along the lines of curvature of the middle surface of the shell.

In the free vibration state, i.e.,  $k_f = \lambda_f = p_{\alpha} = 0$  in equation (6.1), the relation

$$u_{\alpha}(\xi_1, \xi_2, t) = U_{i\alpha}(\xi_1, \xi_2) e^{i\omega_i t} \quad (6.4)$$

exists.

Substitution of equation (6.3) into equation (6.1) and then using equation (6.4) gives

$$\sum_{i=1}^{\infty} Q_i \{U_{i\alpha}\} = \{p_{\alpha}\} \quad (6.5)$$

where

$$Q_i = \rho h \frac{d^2 q_i}{dt^2} + \lambda_f \frac{dq_i}{dt} + (k_f + \rho h \omega_i^2) q_i \quad (6.6)$$

Using the orthogonality condition of the modes of free vibration of an arbitrary shell with a set of prescribed homogeneous boundary conditions [24], the following relation is obtained:



$$\int_S \sum_{\alpha=1}^3 U_{i\alpha} U_{j\alpha} dS = \delta_{ij} \int_S \sum_{\alpha=1}^3 U_{i\alpha}^2 dS \quad (6.7)$$

where  $\delta_{ij}$  is the Kronecker delta and  $S$  denotes integration over the middle surface of the shell.

Multiplication of both sides of equation (6.5) by the row vector  $[U_{j\alpha}]$  and use of equation (6.7), results in the following expression for  $q_i$ :

$$\frac{d^2 q_i}{dt^2} + \frac{\lambda_f}{\rho h} \frac{dq_i}{dt} + \left( \frac{k_f}{\rho h} + \omega_i^2 \right) q_i = \frac{1}{\rho h} Q_i(t) \quad (6.8)$$

$Q_i(t)$  now reads

$$Q_i(t) = \frac{\int_S \sum_{\alpha=1}^3 U_{i\alpha} P_{\alpha} dS}{\int_S \sum_{\alpha=1}^3 U_{i\alpha}^2 dS} \quad (6.9)$$

The complete solution of equation (6.8) for the underdamped case  $(\lambda_f/2\rho h)^2 < k_f/\rho h + \omega_i^2$  is now given by

$$\begin{aligned} q_i(t) = & \exp[-\lambda_f t/2\rho h] (A_i \cos \gamma_i t + B_i \sin \gamma_i t) \\ & + (1/\rho h \gamma_i) \int_0^t Q_i(\tau) \exp[-\lambda_f(t-\tau)/2\rho h] \sin \gamma_i(t-\tau) d\tau \end{aligned} \quad (6.10)$$

where

$$\gamma_1 = \left[ (k_f/\rho h) + \omega_1^2 - (\lambda_f/2\rho h)^2 \right]^{1/2} \quad (6.11)$$

and the arbitrary constants  $A_1$  and  $B_1$  are determined from the initial conditions.

Peterson, Howard, and Philippus [20] have investigated the response of launch vehicles to nonstationary random buffeting excitation. The launch vehicles are treated as simple beams. The general method of procedure presented in Reference [20] is followed and extended to shell problems. The loads on the shell are taken as the following sample functions of distributed nonstationary random processes:

$$\{p_\alpha^{(K)}(\xi_1, \xi_2, t)\} = p_{1\alpha}(\xi_1, \xi_2, t) \{p_{2\alpha}^{(K)}(\xi_1, \xi_2, t)\}^* \quad (6.12)$$

where  $p_{1\alpha}(\xi_1, \xi_2, t)$  is a known deterministic function of  $\xi_1, \xi_2$ , and  $t$ ; and  $p_{2\alpha}^{(K)}(\xi_1, \xi_2, t)$  is an element of a stationary Gaussian random process which has zero mean value, cross power spectral density  $G_{\alpha\alpha}(\omega, \xi_1, \xi_1'; \xi_2, \xi_2')$ , and the cross-correlation function  $\Gamma_{\alpha\alpha}(\tau, \xi_1, \xi_1'; \xi_2, \xi_2')$  for each of the  $p_\alpha$  specified in equation (6.12).

While the stationary random process  $\{p_{2\alpha}^{(K)}(\xi_1, \xi_2, t)\}$  is generated by the excitation of some physical phenomenon, the deterministic function  $p_{1\alpha}(\xi_1, \xi_2, t)$  is governed by the amplitude of the excitation.

For simplicity, let us assume the shell to be initially at rest. Equation (6.10) becomes

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\*{....} represents now the ensemble of each of the elements indicated by the superscript (K).

$$q_i(t) = \frac{1}{\rho h \gamma_i} \int_0^t Q_i(\tau) \exp[-\lambda_f(t-\tau)/2\rho h] \quad (6.13)$$

$$\cdot \sin \gamma_i(t-\tau) d\tau$$

The ensemble of responses can be written now as

$$\{U_\alpha^{(K)}(\xi_1, \xi_2, t)\} = \sum_{i=1}^{\infty} \{q_i^{(K)}(t)\} U_{i\alpha}(\xi_1, \xi_2) \quad (6.14)$$

$$= \sum_{i=1}^{\infty} \frac{U_{i\alpha}(\xi_1, \xi_2)}{\rho h \gamma_i} \int_0^t \{Q_i^{(K)}(\tau)\}$$

$$\cdot \exp[-\lambda_f(t-\tau)/2\rho h] \sin \gamma_i(t-\tau) d\tau$$

and

$$\{Q_i^{(K)}(t)\} = \frac{\int_S \sum_{\alpha=1}^3 U_{i\alpha} \{p_\alpha^{(K)}(\xi_1, \xi_2, t)\} dS}{\int_S \sum_{\alpha=1}^3 U_{i\alpha}^2 dS} \quad (6.15)$$

Using the assumption equation (6.12), equation (6.15) takes the form

$$\{Q_i^{(K)}(t)\} = \frac{\int_S \sum_{\alpha=1}^3 U_{i\alpha} p_{1\alpha}(\xi_1, \xi_2, t) \{p_{2\alpha}^{(K)}(\xi_1, \xi_2, t)\} dS}{\int_S \sum_{\alpha=1}^3 U_{i\alpha}^2 dS} \quad (6.16)$$

Substitution of equation (6.16) into equation (6.14) leads to

$$\begin{aligned}
 \left\{ U_{\alpha}^{(K)}(\xi_1, \xi_2, t) \right\} &= \sum_{i=1}^{\infty} V_{i\alpha} \sum_{\alpha=1}^{\infty} \int_0^t \int_S U_{i\alpha}(\xi_1, \xi_2) \\
 &\cdot p_{1\alpha}(\xi_1, \xi_2, t) \left\{ p_{2\alpha}^{(K)}(\xi_1, \xi_2, t) \right\} \\
 &\cdot \exp [-\lambda_f(t-\tau)/2\rho h] \sin \gamma_i(t-\tau) dS d\tau
 \end{aligned} \tag{6.17}$$

with

$$V_{i\alpha} = \frac{U_{i\alpha}(\xi_1, \xi_2)}{\rho h \gamma_i \int_S \sum_{\alpha=1}^{\infty} U_{i\alpha}^2 dS}$$

We are now ready to calculate the statistical quantities for the shell response.

a. Mean value of response

The mean value of  $\{U_{\alpha}(\xi_1, \xi_2, t)\}$  is an ensemble average over  $U_{\alpha}^{(K)}(\xi_1, \xi_2, t)$ :

$$\left\langle U_{\alpha}^{(K)}(\xi_1, \xi_2, t) \right\rangle = \sum_{i=1}^{\infty} V_{i\alpha} \sum_{\alpha=1}^3 \int_0^t \int_S U_{i\alpha}(\xi_1, \xi_2) \tag{6.18}$$

$$\cdot p_{1\alpha}(\xi_1, \xi_2, t) \left\langle p_{2\alpha}^{(K)}(\xi_1, \xi_2, t) \right\rangle$$

$$\exp[-\lambda_f(t-\tau)/2\rho h] \sin \gamma_1(t-\tau) dS d\tau$$

Since the mean value of  $p_2(\xi_1, \xi_2, t)$  goes to zero by assumption and from equation (6.12),

$$\begin{aligned} \langle p_\alpha^{(K)}(\xi_1, \xi_2, t) \rangle &= p_1(\xi_1, \xi_2, t) \langle p_{2\alpha}^{(K)}(\xi_1, \xi_2, t) \rangle \\ &= 0 \end{aligned} \quad (6.19)$$

Thus,

$$\langle U_\alpha^{(K)}(\xi_1, \xi_2, t) \rangle = 0 \quad (6.20)$$

#### b. Cross-Covariance of Response

Since  $\langle U_\alpha^{(K)}(\xi_1, \xi_2, t) \rangle = 0$ , the cross-covariance of the response must equal to the cross-correlation. By definition

$$\Gamma_{\alpha\beta}(t, t'; \xi_1, \xi_1'; \xi_2, \xi_2') = \langle U_\alpha^{(K)}(\xi_1, \xi_2, t) U_\beta^{(K)}(\xi_1', \xi_2', t') \rangle \quad (6.21)$$

or by using equation (6.17), equation (6.21) reads

$$\Gamma_{\alpha\beta}(t, t'; \xi_1, \xi_1'; \xi_2, \xi_2') = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} v_{i\alpha} v_{j\beta}(\xi_1', \xi_2') \sum_{\alpha=1}^3 \sum_{\beta=1}^3 \quad (6.22)$$

$$\cdot \int_0^t \int_0^{t'} \int_S \int_{S'} U_{i\alpha}(\xi_1, \xi_2) U_{j\beta}(\xi_1', \xi_2')$$

$$p_{1\alpha}(\xi_1, \xi_2, t) p_{1\beta}(\xi_1', \xi_2', t') \langle p_{2\alpha}^{(K)}(\xi_1, \xi_2, t)$$

$$p_{2\beta}^{(K)}(\xi_1', \xi_2', t') \rangle \exp[-\lambda_f(t-\tau+t'-\tau')/2\rho h] \sin \gamma_1(t-\tau)$$

$$\sin \gamma_j(t'-\tau') dS'dSd\tau'd\tau$$

Using the assumption that  $p_{2\alpha}^{(K)}(\xi_1, \xi_2, t)$  is stationary gives

$$p_{2\alpha}^{(K)}(\xi_1, \xi_2, t) p_{2\beta}^{(K)}(\xi_1', \xi_2', t') = \int_0^\infty G(\omega; \xi_1, \xi_1'; \xi_2, \xi_2') \quad (6.23)$$

$$\cos \omega(\tau-\tau') d\omega$$

where  $G(\omega)$  is the power spectrum of  $\{p_{2\alpha}^{(K)}(\xi_1, \xi_2, t)\}$ .

Substitution of equation (6.23) into equation (6.22) gives

$$\Gamma_{\alpha\beta}(t, t'; \xi_1, \xi_1'; \xi_2, \xi_2') = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} v_{i\alpha} v_{j\beta}(\xi_1', \xi_2') \sum_{\alpha=1}^3 \sum_{\beta=1}^3 \quad (6.24)$$

$$\int_0^t \int_0^{t'} \int_S \int_{S'} \int_0^\infty U_{1\alpha}(\xi_1, \xi_2) U_{j\beta}(\xi_1', \xi_2') p_{1\alpha}(\xi_1, \xi_2, t)$$

$$p_{1\beta}(\xi_1', \xi_2', t') G(\omega; \xi_1, \xi_1'; \xi_2, \xi_2') \cos \omega(\tau'-\tau')$$

$$\exp[-\lambda_f(t-\tau+t'-\tau')/2\rho h] \sin \gamma_1(t-\tau) \sin \gamma_j(t'-\tau') d\omega dS'dSd\tau'd\tau$$

Since all physically realizable processes involve power spectra which go to zero for sufficiently high frequencies, the

requirement of physical realizability in the present case is that

$$\int_0^{\infty} G(\omega) d\omega < \infty \quad (6.25)$$

and hence the integrals involved in equation (6.24) are convergent.

The order of integration may then be reversed to give

$$\Gamma_{\alpha\beta}(t, t', \xi_1, \xi_1'; \xi_2, \xi_2') = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} V_{i\alpha} V_{j\alpha}(\xi_1', \xi_2') \sum_{\alpha=1}^3 \sum_{\beta=1}^3 \quad (6.26)$$

$$\cdot \left\{ \int_0^t \int_0^{t'} \int_0^t \int_0^{t'} \left( \int_S U_{i\alpha}(\xi_1, \xi_2) P_{1\alpha}(\xi_1, \xi_2, t) \right. \right. \\ \cdot \left. \left. \int_{S'} U_{j\beta}(\xi_1', \xi_2') P_{1\beta}(\xi_1', \xi_2', t') G(\omega; \xi_1, \xi_1'; \xi_2, \xi_2') dS' \right. \right. \\ \cdot \left. \left. dS \right) \cos \omega(\tau' - \tau) \exp[-\lambda_f(t - \tau + t' - \tau')/2\phi h] \sin \gamma_1(t - \tau) \right. \\ \cdot \left. \sin \gamma_j(t' - \tau') d\tau' d\tau \right) d\omega$$

The double surface integral over  $S$  and  $S'$  resolves the time-dependent CPSD of the excitation into the time-dependent CPSD of the excitation of the  $i$ -th and  $j$ -th modes. The double integral over  $\tau$  and  $\tau'$  then yields the time-dependent CPSD of the responses in the  $i$ -th and  $j$ -th modes. The integration over  $\omega$  gives the cross-covariance of the modal response; and finally, the double summation over the modes gives the cross-covariance of the response\*

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\*See Reference [20].

Consider the particular case when  $p_{1\alpha}(\xi_1, \xi_2, t) = p_{1\alpha}(\xi_1, \xi_2)$ , i.e., when equation (6.12) represents a stationary random process. Furthermore,  $G(\omega, \xi_1, \xi_1'; \xi_2, \xi_2') = G(\omega)$ . For the time-dependent, normalized, cross-correlation function, one has

$$\Gamma_{\alpha\beta}(t, t; \xi_1, \xi_1'; \xi_2, \xi_2') = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} v_{i\alpha} v_{j\beta} \delta_{iK} \delta_{jK} \sum_{\alpha=1}^3 \sum_{\beta=1}^3 \cdot \langle A_{K\alpha} A_{K\beta} \sigma_q^2(t) \rangle \quad (6.27)$$

where

$$p_{1\alpha}(\xi_1, \xi_2) = \sum_{K=1}^{\infty} A_{K\alpha} U_{K\alpha}(\xi_1, \xi_2) \quad (6.28)$$

has been expanded in terms of the orthogonal function  $U_{K\alpha}(\xi_1, \xi_2)$ .  $A_{K\alpha}$  is the normalization factor, and in equation (6.27)

$$\sigma_q^2(t) = \int_0^{\infty} \int_0^t \int_0^t \frac{G(\omega)}{\gamma_1^2} \cos \omega(\tau' - \tau) \exp[-\lambda_f(t - \tau + t' - \tau')/2\rho h] \cdot \sin \gamma_1(t - \tau) \sin \gamma_1(t - \tau') d\tau' d\tau d\omega \quad (6.29)$$

represents the normalized variance of  $q_1(t)$ . Equation (6.29) is particularly interesting, since it is similar to the variance of the harmonic oscillator obtained by Caughey and Stumpf [12]. In fact, the two become identical if we set  $\zeta\omega_0 = \lambda_f/2\rho h$  and  $\omega_1 = \gamma_1$ . Integration of equation (6.29) gives



$$\begin{aligned}
\sigma_q^2(t) = & \int_0^\infty \frac{G(\omega)}{|Z(\omega)|^2} \left[ 1 + \exp[-(\lambda_f/\rho h)t] \left\{ 1 + \frac{\lambda_f/\rho h}{\gamma_1} \right. \right. \\
& \cdot \left. \sin \gamma_1 t \cos \gamma_1 t - \exp[\lambda_f/2\rho h)t] \left\{ 2 \cos \gamma_1 t \right. \right. \\
& + \frac{\lambda_f/\rho h}{\gamma_1} \sin \gamma_1 t \left. \right\} \cos \gamma_1 t - \exp[\lambda_f/2\rho h)t] \frac{2\omega}{\gamma_1} \\
& \cdot \left. \sin \gamma_1 t \sin \omega t + \frac{(\lambda_f/2\rho h)^2 - \gamma_1^2 + \omega^2}{\gamma_1^2} \sin^2 \gamma_1 t \right] d\omega
\end{aligned} \tag{6.30}$$

where

$$|Z(\omega)|^2 = \left[ (\lambda_f/2\rho h)^2 + \omega^2 - \gamma_1^2 \right] + (\omega\lambda_f/\rho h)^2$$

Some of the properties of  $\sigma_q^2(t)$  observed in reference [12] are:

(1) As  $t \rightarrow 0$

$$\sigma_q^2(t) \rightarrow 0$$

(2) As  $t \rightarrow \infty$

$$\sigma_q^2(t) \rightarrow \int_0^\infty \frac{G(\omega)d\omega}{|Z(\omega)|^2}$$

The integration of equation (6.29) may be done analytically, numerically, or approximately, depending on how the power spectrum is given.

Once the geometry of the shell is known, and if the orthogonal functions  $u_{i\alpha}(\xi_1, \xi_2)$  for each  $\alpha$  are given, equation (6.27) is completely

defined by knowing the deterministic function from the input data and the power spectrum for the particular stationary Gaussian random process. Consider as a special example, the hemispherical shell subjected to the presently defined separable nonstationary random process.

### Axisymmetric Response of Hemispherical Shells

Consider that the viscoelastic foundation is absent in the present analysis so that the previously obtained result for the dynamic response of a hemispherical shell may serve the purpose here.

The governing differential equations for the hemispherical shell have been solved in terms of the normal modes from equation (4.31) to give

$$u_{\phi}(x,t) = - \sum_{\substack{n=2,4,\dots \\ \text{or} \\ n=1,3,5,\dots}}^{\infty} \psi_n(t) P_n'(x) \sqrt{1-x^2}$$

$$w(x,t) = \sum_{\substack{n=0,2,4,\dots \\ \text{or} \\ n=1,3,5,\dots}}^{\infty} w_n(t) P_n(x)$$

where  $n = 0, 2, 4, \dots$  corresponds to the roller-clamped boundary case, and  $n = 1, 3, 5, \dots$  corresponds to the roller-hinged boundary case. Prime denotes differentiation with respect to  $x$ , and  $P_n(x)$  is the  $n$ -th degree Legendre polynomial. Referring to equations (4.19), (4.26), and (4.29),  $\psi_n(t)$  and  $w_n(t)$  are the following:

For  $n = 0$ ,

$$w_0 = \frac{A_1}{\lambda \omega_{10}} \int_0^t \sin \omega_{10} (t-\tau) \int_0^1 P_n(\xi, \tau) d\xi d\tau \quad (6.31)$$

$\psi_0$  has been neglected, since it contributes nothing to the displacement  $u_\phi$  due to the fact that  $P_0'(x) = 0$ . Here

$$A_1 = \frac{a^2(1-\nu^2)}{Eh}, \lambda = \rho \frac{a^2(1-\nu^2)}{E}, \omega_{10} = \left[ \frac{2(1-\nu)}{\lambda} \right]^{\frac{1}{2}} \quad (6.32)$$

$a$  is the radius,  $h$  the thickness,  $\rho$  the mass density,  $\nu$  the Poisson's ratio, and  $E$  is the Young's modulus.

For  $n = 1$ ,

$$\begin{aligned} \psi_1(t) = \frac{1}{3\lambda} & \left\{ \frac{1}{\omega_{11}} \int_0^t F_{21}(\tau) \sin \omega_{11}(t-\tau) d\tau \right. \\ & \left. - \int_0^t \int_0^\tau F_{11}(\tau') d\tau' d\tau \right\} \\ w_1(t) = \frac{1}{3\lambda} & \left\{ \frac{2}{\omega_{11}} \int_0^t F_{21}(\tau) \sin \omega_{11}(t-\tau) d\tau \right. \\ & \left. + \int_0^t \int_0^\tau F_{11}(\tau') d\tau' d\tau \right\} \end{aligned} \quad (6.33)$$

where

$$F_{11} = 3A_1 \left[ \int_0^1 P_n(\xi, \tau) P_1(\xi) d\xi - 2 \int_0^1 P_1(\xi) \int_0^{\phi(\xi)} P_\phi(\phi, \tau) d\phi d\xi \right] \quad (6.34)$$

$$F_{21} = 3A_1 \left[ \int_0^1 P_n(\xi, \tau) P_1(\xi) d\xi + \int_0^1 P_1(\xi) \int_0^{\phi(\xi)} P_\phi(\phi, \tau) d\phi d\xi \right]$$

For  $n \geq 2$ ,

$$\begin{aligned} \psi_n(t) = \frac{1}{\lambda^2(\omega_{2n}^2 - \omega_{1n}^2)} & \left\{ \frac{1}{\omega_{1n}} \int_0^t \sin \omega_{1n}(t-\tau) F_{1n}(\tau) d\tau \right. \\ & \left. - \frac{1}{\omega_{2n}} \int_0^t \sin \omega_{2n}(t-\tau) F_{1n}(\tau) d\tau \right\} \\ w_n(t) = \frac{1}{\lambda^2(\omega_{2n}^2 - \omega_{1n}^2)} & \left\{ \frac{1}{\omega_{1n}} \int_0^t \sin \omega_{1n}(t-\tau) F_{2n}(\tau) d\tau \right. \\ & \left. - \frac{1}{\omega_{2n}} \int_0^t \sin \omega_{2n}(t-\tau) F_{2n}(\tau) d\tau \right\} \end{aligned} \quad (6.35)$$

here

$$\begin{aligned} F_{1n}(\tau) = A_1(2n+1) & \left\{ \left( A_{3n} + \lambda \frac{d^2}{d\tau^2} \right) \int_0^1 P_n(\xi) \int_0^{\phi(\xi)} P_\phi(\phi, \tau) d\phi d\xi \right. \\ & \left. - [\alpha \Lambda_n + (1+\nu)] \int_0^1 P_n(\xi, \tau) P_n(\xi) d\xi \right\} \\ F_{2n}(\tau) = A_1(2n+1) & \left\{ \left[ \Lambda_n - (1-\nu) + \lambda \frac{d^2}{d\tau^2} \right] \right. \end{aligned} \quad (6.36a)$$

$$\cdot \int_0^1 p_n(\xi, \tau) p_n(\xi) d\xi + \Lambda_n [\alpha \Lambda_n + (1+\nu)]$$

$$\cdot \int_0^1 p_n(\xi) \int_0^{\phi(\xi)} p_\phi(\xi, \tau) d\phi d\tau >$$

$$\omega_{1n}^2 = \frac{1}{2\lambda} \left[ A_{1n} + (A_{1n}^2 - 4A_{2n})^{\frac{1}{2}} \right] \quad (6.36b)$$

$$\omega_{1n}^2 = \frac{1}{2\lambda} \left[ A_{1n} - (A_{1n}^2 - 4A_{2n})^{\frac{1}{2}} \right]$$

$$\Lambda_n = n(n+1)$$

with

$$A_{1n} = \alpha \Lambda_n [\Lambda_n - (1-\nu)] + \Lambda_n + (1+3\nu)$$

$$A_{2n} = \alpha [\Lambda_n (\Lambda_n - 1+\nu)^2 - \alpha^2 \Lambda_n^3 - 2\alpha \Lambda_n^2 (1+\nu)]$$

$$+ (1-\nu^2) [\Lambda_n - 2]$$

$$A_{3n} = \alpha \Lambda_n (\Lambda_n - 1+\nu) + 2(1+\nu)$$

$$\alpha = \frac{1}{12} \left( \frac{h}{a} \right)^2$$

Equations (6.12) take the form

$$\left\{ p_\phi^{(K)}(\xi, t) \right\} = p_{1\phi}(\xi, t) \left\{ p_{2\phi}^{(K)}(\xi, t) \right\} \quad (6.37)$$

$$\{p_n^{(K)}(\xi, t)\} = p_{1n}(\xi, t) \{p_{2n}^{(K)}(\xi, t)\}$$

for the present case.

The statistical quantities are

a. Mean value of response

Equation (6.20) is now

$$\langle u_\phi^{(K)}(x, t) \rangle = 0 \quad (6.38)$$

$$\langle w^{(K)}(x, t) \rangle = 0$$

which may be obtained directly from equations (6.31) to (6.36).

b. Covariance of Response

The correlation functions for the displacements  $u_\phi(x, t)$  and  $w(x, t)$  are

$$\Gamma_{uu}(t, t'; x, x') = \sum_{\substack{n=2,4,6,\dots \\ \text{or} \\ n=1,3,5,\dots}}^{\infty} \langle \psi_n^{(K)}(t) \psi_n^{(K)}(t') \rangle P_n'(x)^2 \quad (6.39)$$

$$\Gamma_{ww}(t, t'; x, x') = \sum_{\substack{n=0,2,4,\dots \\ \text{or} \\ n=1,3,5,\dots}}^{\infty} \langle w_n^{(K)}(t) w_n^{(K)}(t') \rangle P_n(x)^2$$

$$\Gamma_{uw}(t, t'; x, x') = \sum_{\substack{n=2,4,\dots \\ \text{or} \\ n=1,3,5,\dots}} \langle \psi_n^{(K)}(t) w_n^{(K)}(t') \rangle P_n'(x) P_n(x)$$

The correlation functions  $\langle \psi_n^{(K)}(t) \psi_n^{(K)}(t') \rangle$ ,  $\langle w_n^{(K)}(t) \cdot w_n^{(K)}(t') \rangle$  and  $\langle \psi_n^{(K)}(t) w_n^{(K)}(t') \rangle$  are obtained by using equations (6.32), (6.33), (6.34), and (6.35).

For  $n = 0$ ,

$$\langle w_0^{(K)}(t) w_0^{(K)}(t') \rangle \quad (6.40)$$

$$= \left( \frac{A_1}{\lambda \omega_{10}} \right)^2 \int_0^t \int_0^{t'} \sin \omega_{10}(t-\tau) \sin \omega_{10}(t'-\tau')$$

$$\cdot \int_0^1 \int_0^1 p_{1n}(\xi, \tau) p_{1n}(\xi', \tau') \langle p_{2n}^{(K)}(\xi, \tau) p_{2n}^{(K)}(\xi', \tau') \rangle$$

$$\cdot d\xi' d\xi d\tau' d\tau$$

For  $n = 1$ ,

$$\langle \psi_1^{(K)}(t) \psi_1^{(K)}(t') \rangle \quad (6.41)$$

$$= \frac{1}{(3\lambda)^2} \left[ \frac{1}{\omega_{11}^2} \int_0^t \int_0^{t'} \sin \omega_{11}(t-\tau) \sin \omega_{11}(t'-\tau') \langle F_{21}(\tau) F_{21}(\tau') \rangle \right.$$

$$\cdot d\tau d\tau' - \frac{1}{\omega_{11}} \int_0^t \int_0^{t'} \int_0^{\tau'} \sin \omega_{11}(t-\tau) \langle F_{21}(\tau) F_{11}(\eta') \rangle d\eta' d\tau' d\tau$$

$$- \frac{1}{\omega_{11}} \int_0^t \int_0^t \int_0^{\tau'} \sin \omega_{11}(t'-\tau') \langle F_{21}(\tau') F_{11}(\eta) \rangle d\eta d\tau' d\tau$$

$$+ \int_0^t \int_0^{t'} \int_0^\tau \int_0^{\tau'} \langle F_{11}(\eta) F_{11}(\eta') \rangle d\eta' d\eta d\tau' d\tau \Bigg]$$

$$\langle w_1^{(K)}(t) w_1^{(K)}(t') \rangle$$

$$\begin{aligned} &= \frac{1}{(3\lambda)^2} \left[ \frac{4}{\omega_{11}^2} \int_0^t \int_0^{t'} \text{Sin}\omega_{11}(t-\tau) \text{Sin}\omega_{11}(t'-\tau') \langle F_{21}(\tau) F_{21}(\tau') \rangle \right. \\ &\quad \cdot d\tau' d\tau + \frac{2}{\omega_{11}} \int_0^t \int_0^{t'} \int_0^\tau \text{Sin}\omega_{11}(t-\tau) \langle F_{21}(\tau) F_{11}(\eta') \rangle \\ &\quad \cdot d\eta' d\tau' d\tau + \frac{2}{\omega_{11}} \int_0^t \int_0^{t'} \int_0^\tau \text{Sin}\omega_{11}(t'-\tau') \langle F_{21}(\tau') F_{11}(\eta) \rangle \\ &\quad \left. d\eta d\tau' d\tau + \int_0^t \int_0^{t'} \int_0^\tau \int_0^{\tau'} \langle F_{11}(\eta) F_{11}(\eta') \rangle d\eta' d\eta d\tau' d\tau \right] \end{aligned}$$

where

$$\langle F_{21}(\tau) F_{21}(\tau') \rangle$$

(6.42)

$$\begin{aligned} &= (3A_1)^2 \left[ \int_0^1 \int_0^1 \Gamma_{nn'} P_1(\xi) P_1(\xi') d\xi d\xi' \right. \\ &\quad + \int_0^1 \int_0^1 \int_0^1 \phi'(\xi') \Gamma_{\phi'n} P_1(\xi) P_1(\xi') d\phi d\xi' d\xi \\ &\quad \left. + \int_0^1 \int_0^1 \int_0^1 \phi'(\xi') \int_0^{\phi(\xi)} \Gamma_{\phi'n} P_1(\xi) P_1(\xi') d\phi' d\phi d\xi' d\xi \right] \end{aligned}$$



$$\begin{aligned}
& \langle F_{21}(\tau) F_{11}(\tau') \rangle \\
&= (3A_1)^2 \left[ \int_0^1 \int_0^1 \Gamma_{nn}, P_1(\xi) P_1(\xi') d\xi' d\xi \right. \\
&+ \int_0^1 \int_0^1 \int_0^1 \phi(\xi) \Gamma_{\phi n}, P_1(\xi) P_1(\xi') d\phi d\xi' d\xi \\
&- 2 \int_0^1 \int_0^1 \int_0^1 \phi'(\xi') \Gamma_{\phi', n} P_1(\xi) P_1(\xi') d\phi' d\xi' d\xi \\
&\left. - 2 \int_0^1 \int_0^1 \int_0^1 \phi(\xi) \int_0^1 \phi'(\xi') \Gamma_{\phi n}, P_1(\xi) P_1(\xi') d\phi d\phi' d\xi' d\xi \right]
\end{aligned}$$

$$\begin{aligned}
& \langle F_{11}(\tau) F_{11}(\tau') \rangle \\
&= (3A_1)^2 \left[ \int_0^1 \int_0^1 \Gamma_{nn}, P_1(\xi) P_1(\xi') d\xi d\xi' \right. \\
&- 2 \int_0^1 \int_0^1 \int_0^1 \phi'(\xi') \Gamma_{\phi', n} P_1(\xi) P_1(\xi') d\phi' d\xi' d\xi \\
&- 2 \int_0^1 \int_0^1 \int_0^1 \phi(\xi) \Gamma_{\phi n}, P_1(\xi) P_1(\xi') d\phi d\xi' d\xi \\
&\left. + 4 \int_0^1 \int_0^1 \int_0^1 \phi(\xi) \int_0^1 \phi'(\xi') \Gamma_{\phi n}, P_1(\xi) P_1(\xi') d\phi' d\phi d\xi' d\xi \right]
\end{aligned}$$

with

$$\Gamma_{nn'} = \langle p_n(\xi, t) p_n(\xi', t') \rangle, \quad \Gamma_{\phi'n} = \langle p_\phi(\xi', t) p_n(\xi, t) \rangle$$

$$\Gamma_{\phi n'} = \langle p_\phi(\xi, t) p_n(\xi', t') \rangle$$

For  $n \geq 2$ ,

$$\langle \psi_n^{(K)}(t) \psi_n^{(K)}(t') \rangle \quad (6.43)$$

$$\begin{aligned} &= \frac{1}{\lambda^4(\omega_{2n}^2 - \omega_{1n}^2)} \left\{ \frac{1}{\omega_{1n}^2} \int_0^t \int_0^{t'} \sin \omega_{1n}(t-\tau) \sin \omega_{1n}(t'-\tau') \right. \\ &\quad \cdot \langle F_{1n}^{(K)}(\tau) F_{1n}^{(K)}(\tau') \rangle d\tau' d\tau - \frac{1}{\omega_{1n}\omega_{2n}} \int_0^t \int_0^{t'} \sin \omega_{1n}(t-\tau) \\ &\quad \cdot \sin \omega_{2n}(t'-\tau') \langle F_{1n}^{(K)}(\tau) F_{1n}^{(K)}(\tau') \rangle d\tau' d\tau \\ &\quad - \frac{1}{\omega_{1n}\omega_{2n}} \int_0^t \int_0^{t'} \sin \omega_{2n}(t-\tau) \sin \omega_{1n}(t'-\tau') \langle F_{1n}^{(K)}(\tau) F_{1n}^{(K)}(\tau') \rangle \\ &\quad \cdot d\tau' d\tau + \frac{1}{\omega_{2n}^2} \int_0^t \int_0^{t'} \sin \omega_{2n}(t-\tau) \sin \omega_{2n}(t'-\tau') \\ &\quad \cdot \langle F_{1n}^{(K)}(\tau) F_{1n}^{(K)}(\tau') \rangle d\tau' d\tau \Big\} \end{aligned}$$

$$\langle w_n^{(K)}(t) w_n^{(K)}(t') \rangle$$

$$\begin{aligned}
&= \frac{1}{\lambda^4(\omega_{2n}^2 - \omega_{1n}^2)} \left\{ \frac{1}{\omega_{1n}^2} \int_0^t \int_0^{t'} \sin \omega_{1n}(t-\tau) \sin \omega_{1n}(t'-\tau') \right. \\
&\cdot \left\langle F_{2n}^{(K)}(\tau) F_{2n}^{(K)}(\tau') \right\rangle d\tau' d\tau - \frac{1}{\omega_{1n}\omega_{2n}} \int_0^t \int_0^{t'} \sin \omega_{1n}(t-\tau) \\
&\cdot \sin \omega_{2n}(t'-\tau') \left\langle F_{2n}^{(K)}(\tau) F_{2n}^{(K)}(\tau') \right\rangle d\tau' d\tau \\
&- \frac{1}{\omega_{1n}\omega_{2n}} \int_0^t \int_0^{t'} \sin \omega_{2n}(t-\tau) \sin \omega_{1n}(t'-\tau') \left\langle F_{2n}^{(K)}(\tau) F_{2n}^{(K)}(\tau') \right\rangle \\
&\cdot d\tau' d\tau + \frac{1}{\omega_{2n}^2} \int_0^t \int_0^{t'} \sin \omega_{2n}(t-\tau) \sin \omega_{2n}(t'-\tau') \\
&\cdot \left\langle F_{2n}^{(K)}(\tau) F_{2n}^{(K)}(\tau') \right\rangle d\tau' d\tau \left. \right\}
\end{aligned}$$

The expression for  $\left\langle \psi_n^{(K)}(t) w_n^{(K)}(t') \right\rangle$  can be similarly obtained.

Here

$$\begin{aligned}
&\left\langle F_{1n}^{(K)}(t) F_{1n}^{(K)}(t') \right\rangle \tag{6.44} \\
&= A^{2(2n+1)} \left\{ \int_0^1 \int_0^1 P_n(\xi) P_n(\xi') \int_0^{\phi(\xi)} \int_0^{\phi'(\xi')} \right. \\
&\cdot \left( A_{3n} P_{\phi}^{(K)}(\xi, t) + \lambda \frac{\partial^2 P_{\phi}^{(K)}(\xi, t)}{\partial t^2} \right) \left( A_{3n} P_{\phi}^{(K)}(\xi', t') \right. \\
&\left. \left. + \lambda \frac{\partial^2 P_{\phi}^{(K)}(\xi', t')}{\partial t'^2} \right) d\phi' d\phi d\xi' d\xi - [\alpha \Lambda_n + (1+\nu)] \int_0^1 \int_0^1 P_n(\xi) P_n(\xi') \right.
\end{aligned}$$

$$\cdot \int_0^{\phi(\xi)} \left\langle A_{3n} p_{\phi}^{(K)}(\xi, t) + \lambda \frac{\partial^2 p_{\phi}^{(K)}(\xi, t)}{\partial t^2} \right\rangle p_n^{(K)}(\xi', t') \rangle$$

$$\cdot d\phi d\xi' d\xi - [\alpha \Lambda_n + (1+\nu)] \int_0^1 \int_0^1 \int_0^1 \phi'(\xi') p_n(\xi) p_n(\xi')$$

$$\cdot \left\langle p_n^{(K)}(\xi, t) \left( A_{3n} p_{\phi}^{(K)}(\xi', t') + \lambda \frac{\partial^2 p_{\phi}^{(K)}(\xi', t')}{\partial t'^2} \right) \right\rangle d\phi' d\xi' d\xi$$

$$+ [\alpha \Lambda_n + (1+\nu)] \int_0^1 \int_0^1 \int_0^1 \phi'(\xi') \langle p_n^{(K)}(\xi, t) p_n^{(K)}(\xi', t') \rangle d\phi' d\xi' d\xi \Bigg\}$$

$$\langle F_{2n}^{(K)}(t) F_{2n}^{(K)}(t') \rangle$$

$$= A^2 (2n+1)^2 \left\{ \int_0^1 \int_0^1 p_n(\xi) p_n(\xi') \left\langle \left[ \Lambda_n - (1-\nu) + \lambda \frac{\partial^2}{\partial t^2} \right] p_n^{(K)}(\xi, t) \right. \right.$$

$$\cdot \left. \left[ \Lambda_n - (1-\nu) + \lambda \frac{\partial^2}{\partial t'^2} \right] p_n^{(K)}(\xi', t') \right\rangle d\xi' d\xi + \int_0^1 \int_0^1 \int_0^1 \phi'(\xi')$$

$$\cdot p_n(\xi) p_n(\xi') \left\langle \left[ \Lambda_n - (1-\nu) + \lambda \frac{\partial^2}{\partial t^2} \right] p_n^{(K)}(\xi, t) \right. \Lambda_n [\alpha \Lambda_n + (1+\nu)]$$

$$\cdot p_{\phi}^{(K)}(\xi', t') \left. d\phi' d\xi' d\xi' + \int_0^1 \int_0^1 \int_0^1 \phi(\xi) p_n(\xi) p_n(\xi') \right.$$

$$\cdot \left. \Lambda_n [\alpha \Lambda_n + (1+\nu)] p_{\phi}^{(K)}(\xi, t) \left[ \Lambda_n - (1-\nu) + \lambda \frac{\partial^2}{\partial t'^2} \right] \right.$$

$$\cdot p_n^{(K)}(\xi', t') \rangle d\phi d\xi' d\xi + \Lambda_n^2 [\alpha \Lambda_n + (1+\nu)] \int_0^1 \int_0^1 \int_0^1 \phi(\xi) \phi'(\xi')$$

$$\cdot p_n(\xi) p_n(\xi') \langle p_\phi^{(K)}(\xi, t) p_\phi^{(K)}(\xi', t') \rangle d\phi d\xi' d\xi$$

Since the substitution of equations (6.37) into equations (6.44) leads to rather lengthy expressions, we shall consider equations (6.39), (6.40), (6.41), (6.42), (6.43), and (6.44) to be complete for the present purpose.

#### Numerical Example

For simplicity, let the roller-clamped hemispherical shell be subjected to a uniformly distributed nonstationary random process of the following form:

$$\begin{aligned} \{ p_\phi^{(K)}(x, t) \} &= 0, \quad 0 \leq t \leq T \\ \{ p_n^{(K)}(x, t) \} &= p_{1n}(t) \{ p_{2n}^{(K)}(t) \}, \quad 0 \leq t \leq T \end{aligned} \quad (6.45)$$

Equations (6.45) indicate that the random process is uniformly distributed over the shell surface, and perpendicular to the middle surface of the shell only. One such application for this dynamic model is that of the hemispherical nose of a space vehicle subjected to the transonic buffeting pressure.

Knowing that

$$P_0(x) = 1 \quad (6.46)$$

one sees immediately that only the breathing mode is excited by this process.

The covariances of the response are

$$\langle u_\phi^{(K)}(x,t) u_\phi^{(K)}(x',t') \rangle = 0 \quad (6.47)$$

$$\begin{aligned} & \langle w^{(K)}(x,t) w^{(K)}(x',t') \rangle \\ &= \langle w_0^{(K)}(t) w_0^{(K)}(t') \rangle \\ &= \left( \frac{A_1}{\lambda \omega_{10}} \right)^2 \int_0^t \int_0^{t'} \sin \omega_{10}(t-\tau) \sin \omega_{10}(t'-\tau') \\ & \quad \cdot p_{1n}(\tau) p_{1n}(\tau') \langle p_{2w}^{(K)}(\tau) p_{2w}^{(K)}(\tau') \rangle d\tau' d\tau \end{aligned}$$

Using the assumption that the random process is uniformly distributed over the shell, equation (6.23) gives

$$\langle p_{2n}^{(K)}(\tau) p_{2n}^{(K)}(\tau') \rangle = \int_0^\infty G(\omega) \cos \omega(\tau-\tau') d\omega \quad (6.48)$$

where  $G(\omega)$  is the power spectrum of  $\{p_{2n}^{(K)}(t)\}$ .

Substitution of equation (6.48) into the second equation of (6.47) gives

$$\langle w^{(K)}(x,t) w^{(K)}(x',t') \rangle \quad (6.49)$$

$$= \left( \frac{A_1}{\lambda \omega_{10}} \right)^2 \int_0^t \int_0^{t'} \int_0^\infty \sin \omega_{10}(t-\tau) \sin \omega_{10}(t'-\tau') \\ \cdot p_{1n}(\tau) p_{1n}(\tau') G(\omega) \cos \omega(\tau-\tau') d\omega d\tau' d\tau$$

Let the excitation  $p_{2n}(x,t)$  be of the following two different types:

a. A uniformly distributed broad-band white noise

$$G(\omega) = \begin{cases} \frac{2D}{\pi} & , \omega \geq 0 \\ 0 & , \omega < 0 \end{cases} \quad (6.50)$$

Equation (6.49) takes the form:

$$\Gamma_{ww}(t, t'; x, x') \quad (6.51) \\ = \frac{DA_1}{(\lambda \omega_{10})^2} \left\{ \cos \omega_{10}(t'-t) \int_0^t p_{1n}^2(\tau) d\tau \right. \\ - \cos \omega_{10}(t'+t) \int_0^t p_{1n}^2(\tau) \cos 2\omega_{10}\tau d\tau \\ \left. - \sin \omega_{10}(t'+t) \int_0^t p_{1n}^2(\tau) \sin 2\omega_{10}\tau d\tau \right\}$$

It is assumed here that  $t' > t$ . For  $t' = t$ , one has the mean-square value of response

$$\sigma_w^2(t) = \frac{DA_1^2}{(\lambda\omega_{10})^2} \left\{ \int_0^t p_{1n}^2(\tau) d\tau \right. \\ \left. - \cos 2\omega_{10}t \int_0^t p_{1n}^2(\tau) \cos 2\omega_{10}\tau d\tau \right. \\ \left. - \sin 2\omega_{10}t \int_0^t p_{1n}^2(\tau) \sin \omega_{10}\tau d\tau \right\} \quad (6.52)$$

b. A uniformly distributed band-limited white noise

$$G(\omega) = \begin{cases} G_0, & -2\omega_{10} < \omega < 2\omega_{10} \\ 0, & |\omega| > 2\omega_{10} \end{cases} \quad (6.53)$$

Equation (6.47) becomes

$$\Gamma_{ww}(t, t'; x, x') \quad (6.54) \\ = \left( \frac{A_1}{\lambda\omega_{10}} \right)^2 G_0 \left\{ \int_0^t \int_0^{t'} \sin \omega_{10}(t-\tau) \sin \omega_{10}(t'-\tau') \right. \\ \left. \cdot p_{1n}(\tau) p_{1n}(\tau') \frac{\sin[2\omega_{10}(t'-t)]}{t'-t} d\tau d\tau' \right\}$$

Again, it has assumed that  $t' > t$ . For  $t' = t$ , the mean-square value of the response is

$$\sigma_w^2(t) = \left( \frac{A_1}{\lambda\omega_{10}} \right)^2 2G_0\omega_{10} \int_0^t \int_0^t [\sin \omega_{10}(t-\tau)]^2 [p_{1n}(\tau)]^2 d\tau d\tau \quad (6.55)$$



Consider the mean-square response with the above two different types of power spectra accompanied with the following forms of deterministic functions:

a. Sinusoidal modulation:

$$p_{1n}(t) = \sin \alpha \omega_0 t, \quad 0 \leq t \leq T \quad (6.56)$$

Case a

$$\sigma_w^2(t) = \frac{DA_1^2}{2(\lambda \omega_{10})^2} \left\{ t + \sin 2\omega_{10} t \frac{\alpha^2 \omega_0^2}{2\omega_{10}(\omega_{10}^2 - \alpha^2 \omega_0^2)} - \sin 2\alpha \omega_0 t \frac{\omega_{10}^2}{2\alpha \omega_0(\omega_{10}^2 - \alpha^2 \omega_0^2)} \right\} \quad (6.57)$$

Case b

$$\sigma_w^2(t) = \left( \frac{A_1}{\lambda \omega_{10}} \right)^2 2G_0 \omega_{10} \left\{ \frac{\omega_{10}}{-\alpha^2 \omega_0^2 + \omega_{10}^2} \sin \alpha \omega_0 t - \frac{\alpha \omega_0}{\omega_{10}^2 - \alpha^2 \omega_0^2} \sin \omega_{10} t \right\}^2 \quad (6.58)$$

b. Constant modulation:

$$p_{1n}(t) = \alpha \omega_0, \quad 0 \leq t \leq T \quad (6.59)$$

Case a

$$\sigma_w^2(t) = \frac{DA_1^2 \alpha^2 \omega_0^2}{(\lambda \omega_{10})^2} \left[ t - \frac{\sin 2\omega_{10}t}{2\omega_{10}} \right] \quad (6.60)$$

Case b

$$\sigma_w^2(t) = \left( \frac{A_1}{\lambda \omega_{10}} \right)^2 2G_0 \omega_{10} \frac{\alpha^2 \omega_0^2}{\omega_{10}^2} [\cos \omega_{10}t - 1]^2 \quad (6.61)$$

The above obtained mean-square responses have the following common properties:

(1) as  $t \rightarrow 0$

$$\sigma_w^2 \rightarrow 0$$

(2) as  $t \rightarrow 0$

$$\frac{\partial}{\partial t} (\sigma_w^2) \rightarrow 0$$

Note here that the root-mean-square response, which is denoted by RMS response, is the square root value of each of the mean square responses.

Numerical results are plotted in Figures 19 through 22 by using an electronic computer with the dimensions and material properties of the shell as

$$\nu = \frac{1}{3}, \quad E = 30 \times 10^6, \quad a = 100 \text{ in.}$$

### Conclusions

(A) Random excitation of thin elastic shells has been studied by using a modal analysis. Numerical results are obtained for

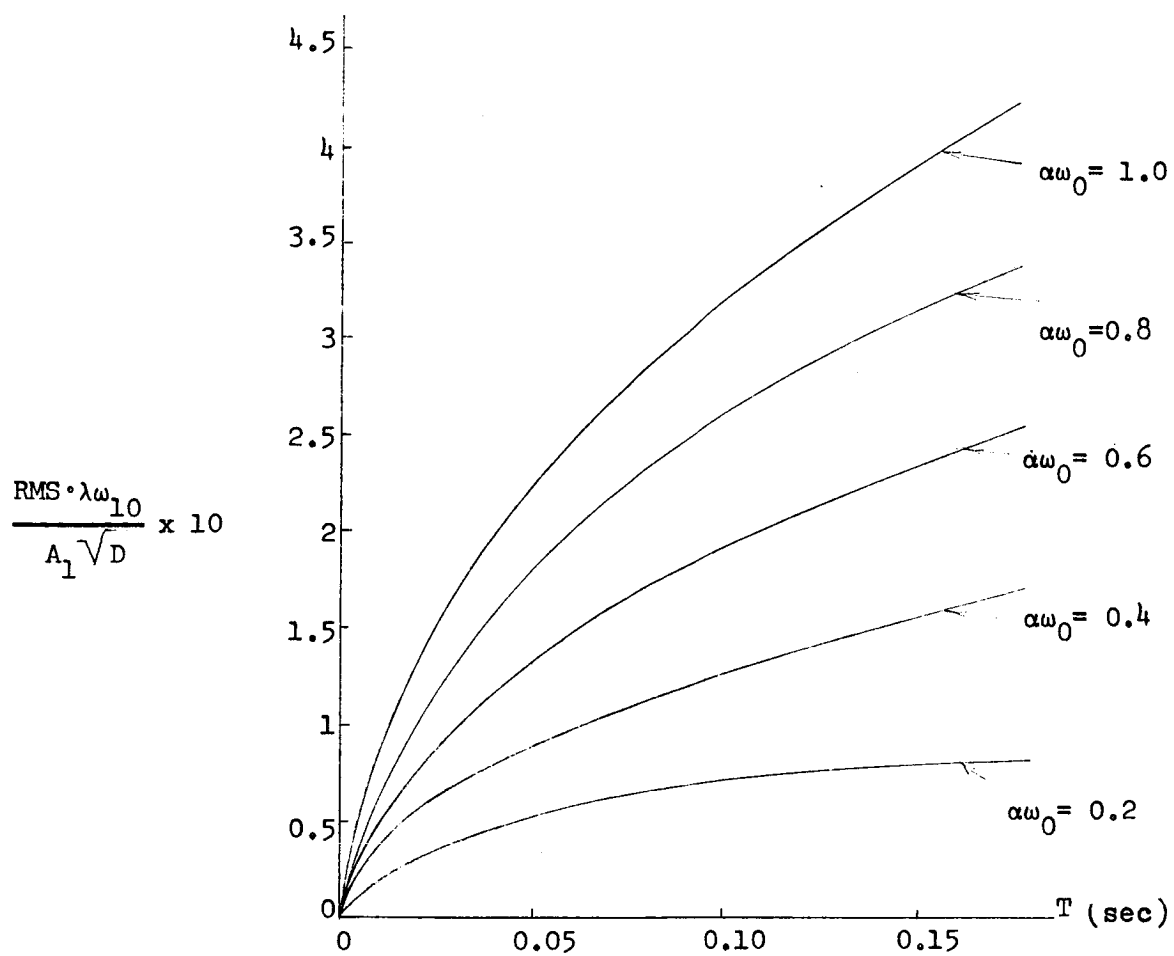


Figure 19. RMS Response, Wide-Band

White Noise,  $p_{1n} = \alpha\omega_0$

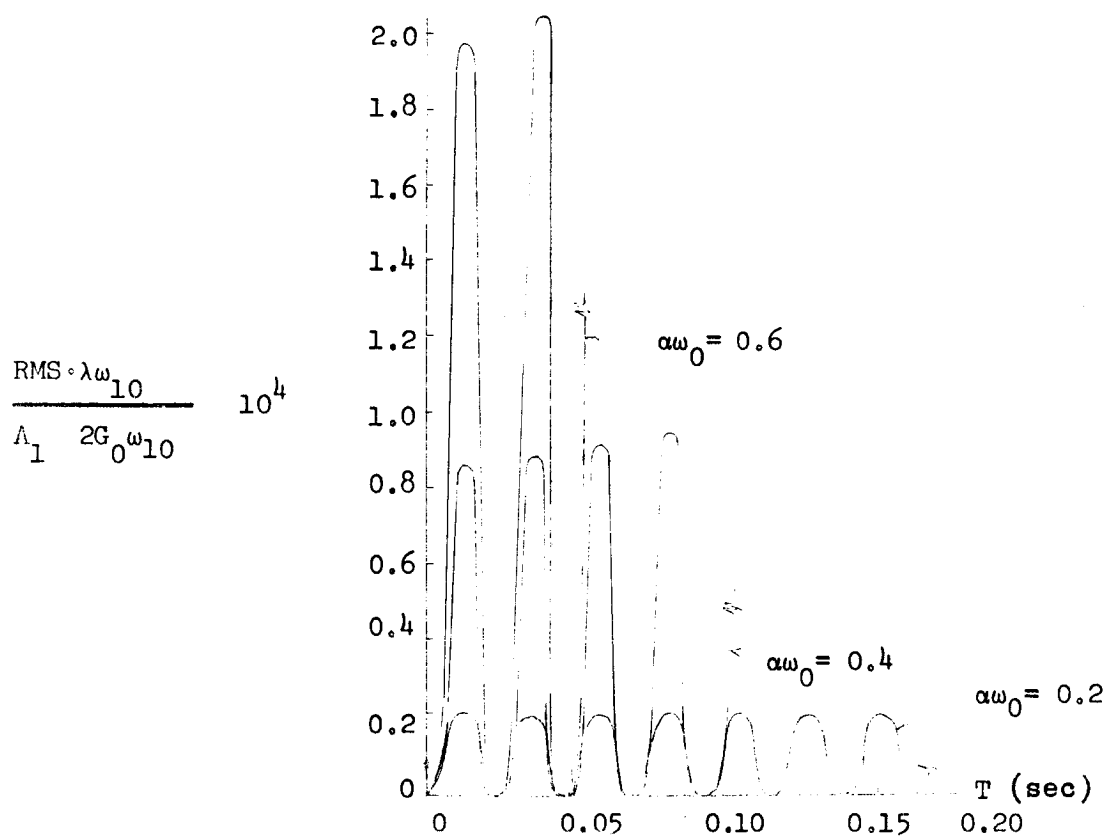


Figure 20. RMS Response, Band-Limited

White Noise,  $p_{1n} = \alpha\omega_0$

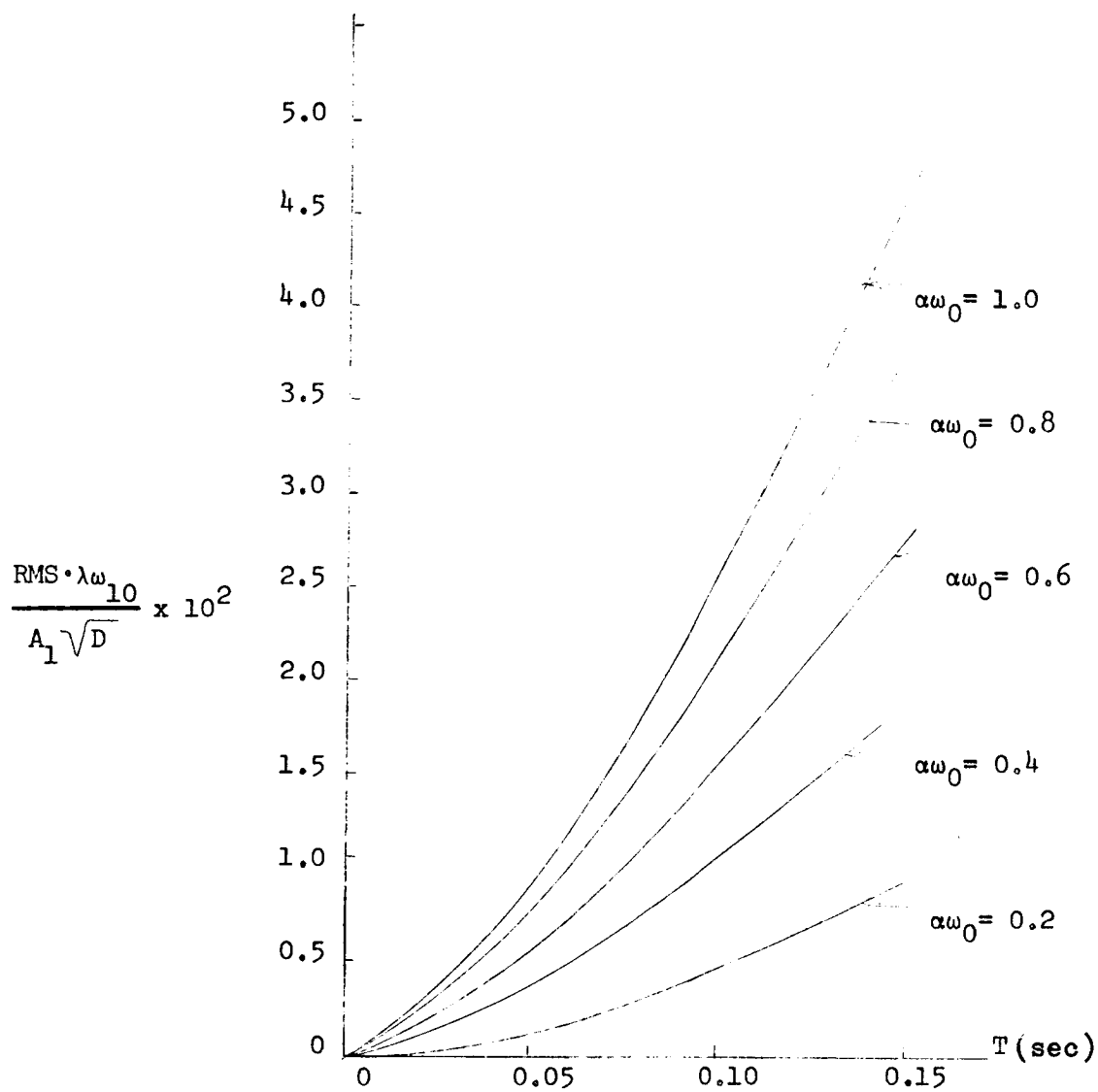


Figure 21. RMS Response, Wide-Band

White Noise,  $p_{1n} = \sin\alpha\omega_0 T$

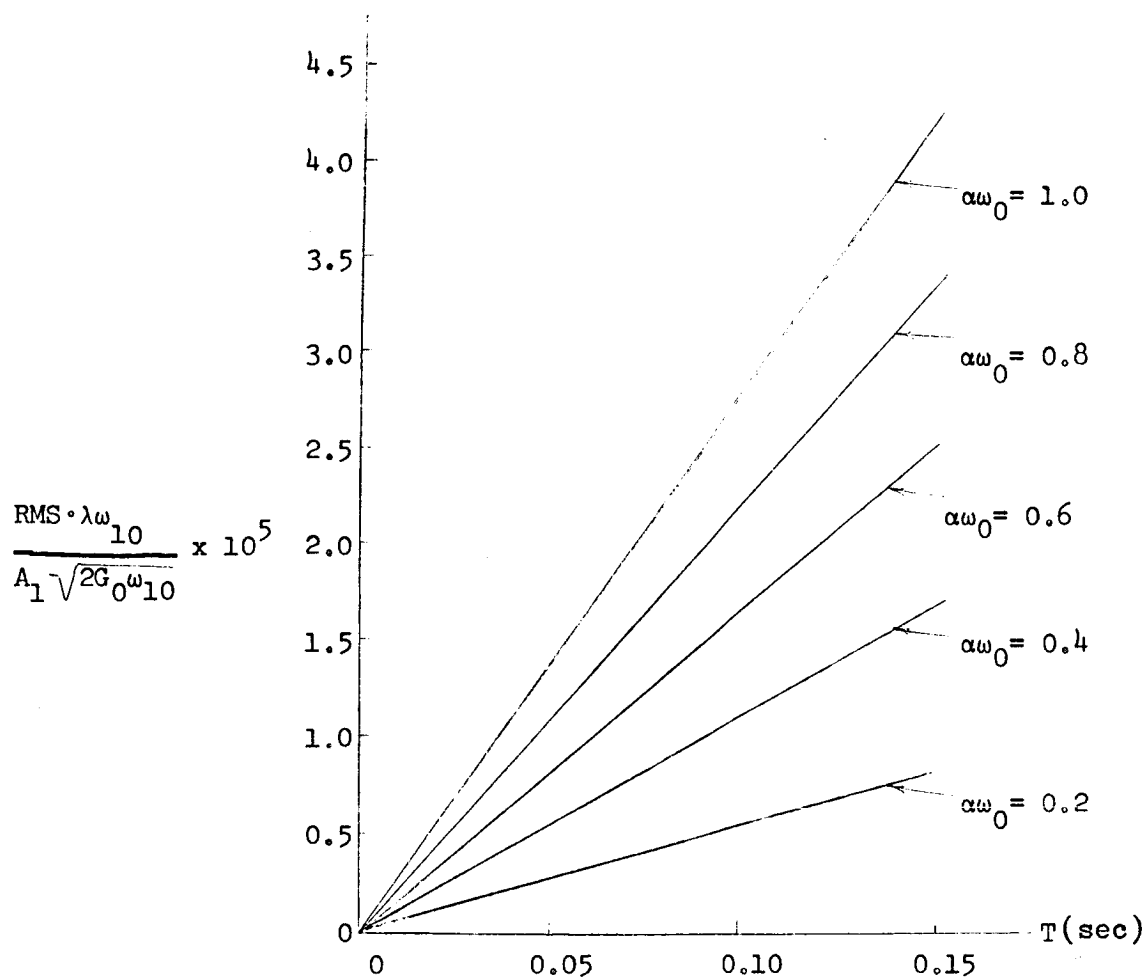


Figure 22. RMS Response, Band-Limited

White Noise,  $p_{1n} = \sin \alpha \omega_0 T$

hemispherical shells with roller-clamped edges subjected to non-stationary separable random processes uniformly distributed over the shell surface. The white noise spectral density applied in the calculation is to simulate the load on a hemispherical shell nose of a space vehicle when experiencing the transonic buffeting pressure during the flight. For this type of hammerhead payload, flight data from a Atlas-Able V model shows that the fluctuating pressures tend to produce nearly constant power spectral densities over a low-frequency region. However, in order to provide a general view, the wide band distribution of the PSD is also included in the study.

(B) In an actual dynamic loads problem, the choice of the deterministic function depends on knowledge of the actual excitation of the vehicle. This knowledge would come from wind tunnel data, flight measurements, an estimate based on practical experience, or a combination of these. The analysis carried out for a hemispherical shell gives one a better insight into the shell type of problem and provides one with a general view of the effect of different types of deterministic functions.

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